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OPTIMUM SIGNAL DESIGN AND PROCESSING FOR REVERBERATION - LIMITED ENVIRONMENTS

By

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PREFACE

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I. SUMMARY

A. PURPOSE AND SCOPE

In this report we discuss the problem of designing optimum signals and receivers for an active sonar system operating in an environment in which the principal cause of interference is reverberation. We have considered only the problem of target detection (as opposed to the problem of range or Doppler estimation).

In many sonar systems the primary limitation on performance is energy scattered back to the receiver from various objects in the ocean. If the scattering structure were known in detail, the reverberation could be treated as a deterministic return and the signal and receiver designed to eliminate it. Because of variations in the ocean, however, this type of model quickly becomes unrealistic or unmanageable. For this reason, we chose an alternate approach in which the reverberation return is treated as a random process. Characterizing the reverberation as such a process enabled us to consider a large number of scatterers efficiently and to attempt to design signals and receivers which will work well on the average.

First, we constructed a suitable statistical model of the random reverberation return. Starting from physical considerations, we showed that the return can be reasonably characterized as a zero-mean non-stationary Gaussian random process. The correlation function of the process, which is a function of the transmitted signal and distribution of the scatterers in range, is assumed to be known.

Once this model was obtained, the conceptual path to the optimum receiver was clear, and we developed the equations which specify the optimum receiver. The parameters in the optimum receiver depend on the structure of the reverberation. The resulting receiver may be complex; frequently, therefore, a simpler, sub-optimum receiver is used, e.g., one which ignores the reverberation and assumes the additive noise is white. In this case, the receiver consists of a filter whose impulse response is the signal reversed in time, a detector, and a threshold device. This type of receiver is a conventional matched filter receiver. We have given expressions for the performance of the optimum and the conventional receiver for arbitary signal shapes and reverberation characteristics.

We also considered the signal design problem. We derived some general properties that are useful in choosing a suitable form for the transmitted signal.

To obtain a quantitative indication of the performance levels that can be achieved using various signal shapes and optimum or conventional receivers, we considered two cases in detail: (1) reverberation which is homogeneous in range and results in interference which is a stationary Gaussian process, (2) reverberation which is non-homogeneous in range and results in a non-stationary Gaussian process. In both cases we were concerned principally with a transmitted signal having a Gaussian envelope and linear frequency modulation.

B. CONCLUSIONS AND RECOMMENDATIONS

1. The most effective way (within the limitations of our model) to combat reverberation is through proper signal design. In fact, proper signal design is more important than optimum receiver design.

2. The evaluation of the performance of the conventional receiver for arbitrary signal shapes and reverberation characteristics is a straightforward calculation. Finding the optimum receiver for arbitrary signals, however, appears both difficult and unrewarding.

3. In many cases, the best signal will not be a Gaussian pulse. The performance achieved using pseudo-random waveforms and simple pulse trains should be evaluated. (See References 14 and 15 for some examples.)

4. In some cases, the optimum receiver will offer enough improvement to warrant its complexity. To implement it, one needs to know the statistical structure of the reverberation and thus needs some means of measurement. Considerable research has been devoted to the related measurement problem in a radar astronomy and communications context. Presumably, some of these results should be adaptable to the reverberation problem.

5. In most cases, the scattering function of the reverberation will vary slowly with time. An ideal system should have provision for continually measuring the scattering function and adapting the transmitted signal shape and receiver to the current environment.

6. We have considered only detectability and time processing. The problems of parameter measurement (such as range and Doppler) and the space-time problem should also be considered.

II. INTRODUCTION

In the simplest detection problem, the signal returned from the target is completely known. In this case, the receiver has available for processing a waveform which consists of either ambient noise, if no target is present, or of ambient noise plus the target return signal if a target is present. The detection problem consists solely of deciding which of the two alternatives is correct. The simplest example of this case occurs when the ambient noise is a sample function from a white, Gaussian random process (spectral height $\frac{N_0}{2}$ volts^2/cps) which is independent from the signal. It is well known/ that in this case, the optimum processor is either a correlation receiver or a matched filter receiver and that the performance depends only on the ratio E/N_0 (where E is the energy in the signal).

Two characteristics of this solution are of interest to us. First, the performance is completely independent of the signal shape. Any signal with a given amount of energy is as good as any other. Second, one can achieve any desired performance level by increasing the transmitted signal energy to a large enough value. We shall see, however, that this simple model does not adequately describe the active sonar problem and that these two characteristics cannot be achieved in an actual situation.

When a signal is transmitted into the ocean, it encounters various inhomogeneities in the medium and numerous objects which cause it to be scattered. The return from these various sources is called reverberation. (In Section III, we shall construct a quantitative model of this reverberation return.) Since the reverberation return is caused by the signal, it is clear that the statistical characteristics of the reverberation "noise" are not independent of the signal shape. One would suspect, therefore, that the receiver performance will no longer be independent of the signal shape. Moreover, increasing the transmitted signal power will increase the level of the reverberation return. Thus, increasing the transmitted energy may be an inefficient way to combat reverberation.

The problem here is one of target detection in the presence of interference which depends on the transmitted signal. The basic ideas involved in our solution are reasonably straightforward, but the manipulations necessary to obtain a quantitative solution are somewhat involved. To illustrate some of the basic concepts involved, we will consider a simple example of target detection in the presence of interference. The model does not represent a realistic sonar problem, but is only a tutorial example.

/We assume that the reader is familiar with the application of statistical detection theory to receivers operating in an additive Gaussian noise environment.

A. AN EXAMPLE OF TARGET DETECTION IN THE PRESENCE OF INTERFERENCE /

Consider the simple problem shown in Figure 1. The signal returned from the target is $S_d(t)$. The signal returned from the interfering object is $S_I(t)$. In addition, an additive, ambient noise n(t) is present. The received waveform is a sum of these three terms.

$$r(t) = S_d(t) + S_I(t) + n(t) \qquad \text{(II-1)}$$

As a special case, let us assume that $S_I(t)$ is identical to $S_d(t)$ except for a time delay and an attenuation.

Thus,

$$S_I(t) = a\, S_d(t - \tau) \qquad \text{(II-2)}$$

where τ is assumed known. If, in addition, the value of "a" is known, the solution is simple. One subtracts out the interfering signal and then uses the usual matched filter receiver. The receiver structure for this simple case is shown in Figure 2. To make the problem more realistic, we must include some uncertainty to the interfering signal. Therefore, we assume that the attenuation is a Gaussian random variable with zero-mean and variance, σ_a^2 .

We now have a familiar two-hypothesis problem.

Under H_0, no signal present, the mean of r(t) is zero and the covariance function is:

$$R(t,u) = E\,[r(t)r(u)] = \frac{N_o}{2}\,\delta(t-u) + \sigma_a^2\, S_d(t-\tau)S_d(u-\tau) \qquad \text{(II-3)}$$

where we assume the additive noise is a sample function from a Gaussian process.

Under H_1, signal present, the mean of r(t) is $S_d(t)$ and the covariance function is unchanged.

/ This example is based on a similar example in an unpublished memorandum by W. M. Siebert.

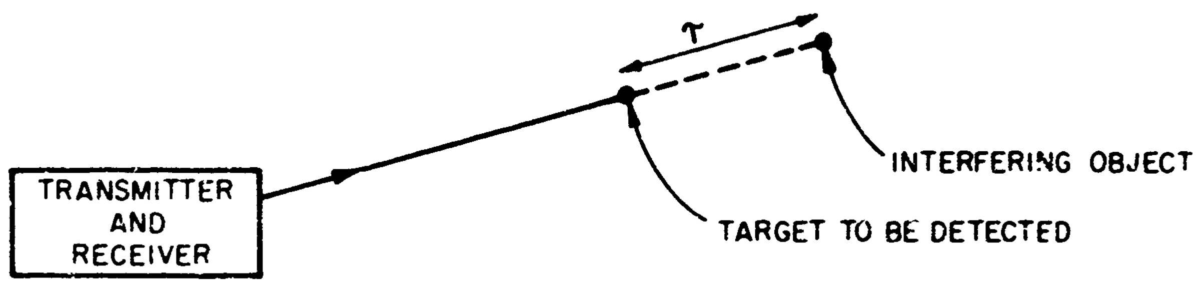

FIGURE 1 GEOMETRY FOR INTERFERENCE EXAMPLE

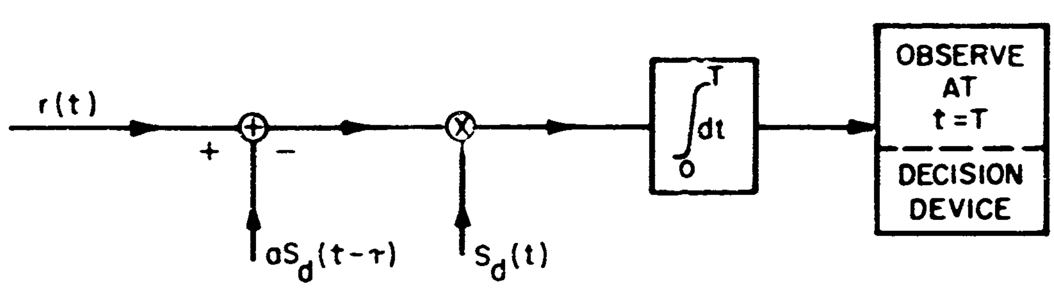

FIGURE 2 OPTIMUM RECEIVER-INTERFERING SIGNAL KNOWN COMPLETELY

The optimum detector consists of a correlation operation on the received signal r(t)

$$X = \int_0^T r(t)\, q(t)dt \tag{II-4}$$

where q(t) is the solution to the integral equation:

$$q(t) = \int_0^T Q(t,u) S_d(u)\, du \qquad 0 < t < T \tag{II-5}$$

and the function Q(t, u) is the "inverse kernel" and satisfied the equation:

$$\delta(t-z) = \int_0^T Q(t,u)\, R(u,z)\, du \tag{II-6}$$

This solution is just a special case of the nonwhite noise problem.

One can verify by direct substitution into Equation II-6 that

$$Q(t,u) = \frac{2}{N_o}\, \delta(t-u) - \frac{2}{N_o} \frac{\sigma_a^2}{\frac{N_o}{2} + \sigma_a^2 E} S_d(t-\tau) S_d(u-\tau) \tag{II-7}$$

Then substituting Equation II-7 into Equation II-5, we obtain:

$$q(t) = \frac{2}{N_o} S_d(t) - \frac{2}{N_o} \frac{\sigma_a^2}{\sigma_a^2 E + \frac{N_o}{2}} S_d(t-\tau) \left[\int_0^T S_d(u) S_d(u-\tau)\, du \right]$$

$$= \frac{2}{N_o} \left[S_d(t) - E\, \gamma_{di}(\tau) \frac{\sigma_a^2}{\sigma_a^2 E + \frac{N_o}{2}} S_d(t-\tau) \right] \tag{II-8}$$

where

$$\gamma_{di}(\tau) \equiv \frac{\int S_d(u) S_d(u-\tau) du}{E} \qquad \text{(II-9)}$$

represents a normalized correlation between the desired signal and the interfering signal. Clearly, $0 \leq \gamma_{di}(\tau) \leq 1$.

The optimum receiver consists of two parts, as shown in Figure 3. One part is the usual correlation operation; the second part is a partial subtraction of the interfering signal.

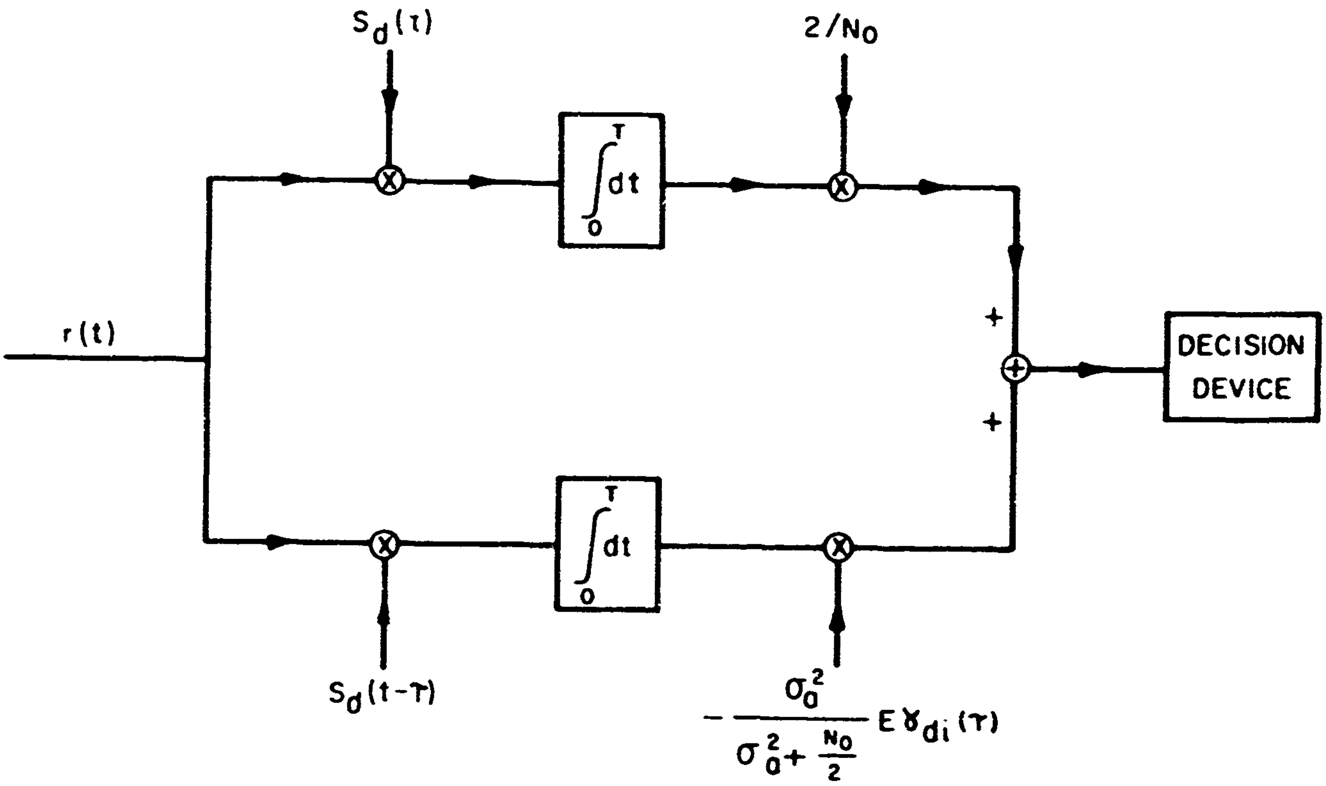

FIGURE 3 OPTIMUM RECEIVER: INTERFERING TARGET WITH RANDOM AMPLITUDE

The optimum receiver has a very good intuitive interpretation. In Figure 2, we saw that if the interfering signal were known exactly, we could subtract it out. A logical approach in the presence of some uncertainty might be to look at the received waveform r(t) and estimate what the interfering signal is. We could then subtract this estimate and pass the result into the normal white noise detector. We will now demonstrate that this logical approach is exactly what the optimum receiver is doing.

The only unknown quantity in the interfering signal is the amplitude "a". One can show that if the received waveform is

$$r(t) = aS_d(t-\tau) + n(t), \qquad \text{(II-10)}$$

then the most probable value of "a" and the minimum variance estimate of "a" are identical. The estimate, $\hat{a}$, is given by

$$\hat{a} = \frac{\sigma_a^2}{\frac{N_o}{2} + \sigma_a^2 E} \int_0^T r(u)\, S_d(u-\tau)\, du \qquad \text{(II-11)}$$

The equation describing the receiver shown in Figure 4 is:

$$X = \int_0^T \left[r(t) - \hat{a}\, S_d(t-\tau) \right] S_d(t)\, dt \qquad \text{(II-12)}$$

Substituting Equation II-11 into Equation II-12, we obtain:

$$X = \frac{2}{N_o} \int_0^T r(t) S_d(t) dt - \frac{2}{N_o} \frac{\sigma_a^2 \, \gamma_{di}(\tau)}{\frac{N_o}{2} + \sigma_a^2 E} \cdot \int_0^T r(u) S_d(u-\tau) du \qquad \text{(II-13)}$$

We see that the two receivers are equivalent. Thus, the optimum receiver does exactly what one might expect.

A complete measure of performance is the ratio of the square of the mean of X under hypothesis H_1 to the variance of X.

Thus.

$$d_o^2 \equiv \frac{\left\{ E\left[X : H_1 \right] \right\}^2}{\text{Var } X} \qquad \text{(II-14)}$$

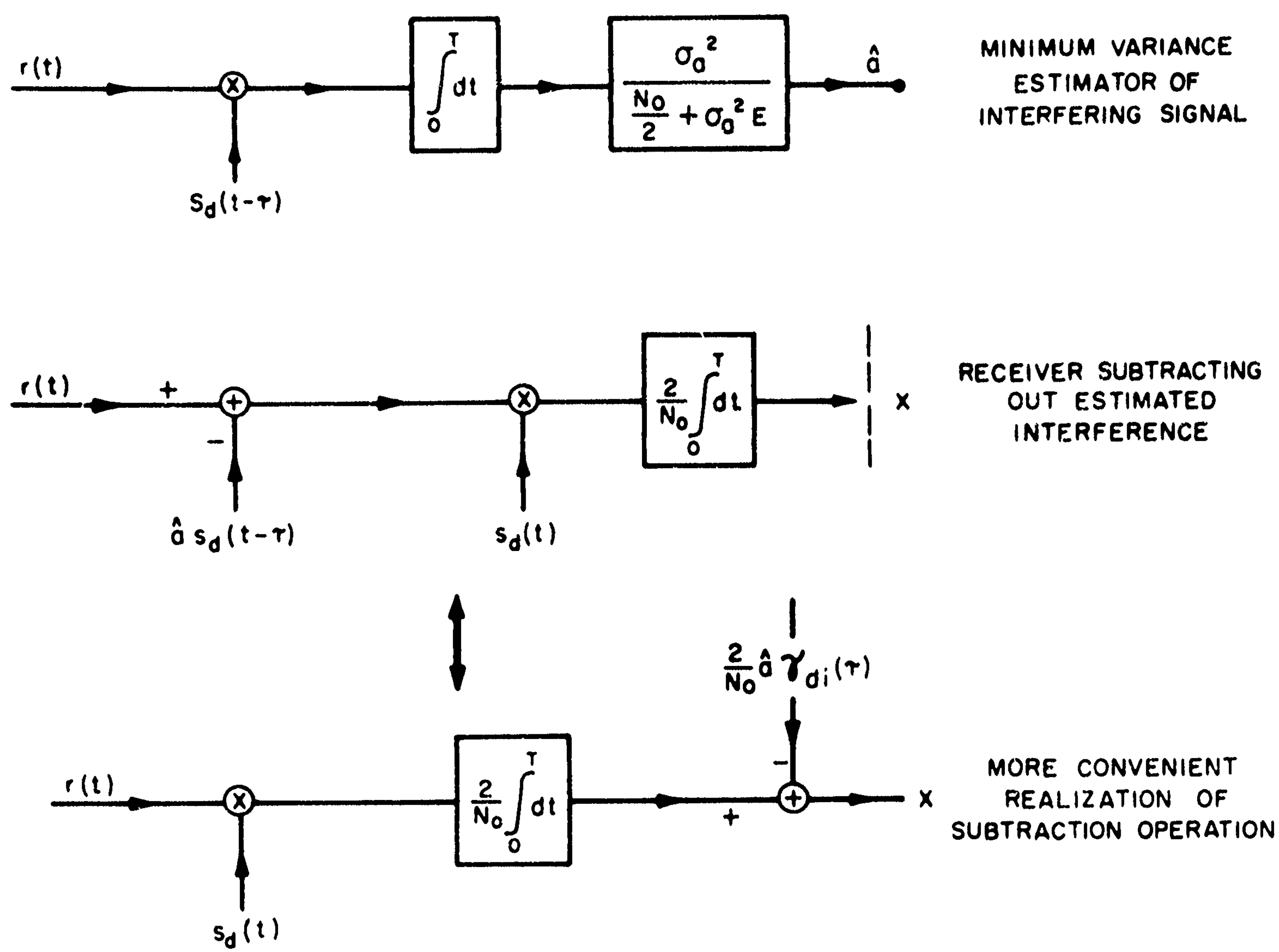

FIGURE 4 ESTIMATOR-SUBTRACTOR RECEIVER

One can show that

$$d_o^2 = \frac{2E}{N_o} \left\{ 1 - \gamma_{di}^2(\tau) \frac{\frac{2E}{N_o} \sigma_a^2}{1 + \frac{2E}{N_o} \sigma_a^2} \right\} \tag{II-15}$$

We observe that the first term is simply the usual white noise result. The second term represents the degradation due to the interfering signal. The magnitude of this degradation depends on:

1. $\gamma_{di}^2(\tau)$: the correlation (or similarity) between the desired signal and the interfering signal.

2. σ_a^2 : the strength of the interfering signal

3. $\frac{2E}{N_o}$: the energy-to-noise-density ratio.

If any of these are small, the effect of the interfering signal will be small. The signal design problem in this case is simply choosing a signal shape so that $\gamma_{di}^2(\tau)$ is small. For fixed τ and no constraint on peak transmitter power, the solution is simple.

Let

$$S_d(t) = \begin{cases} \frac{2\sqrt{E}}{\tau} & 0 < t \leq \frac{\tau}{2} \\ 0 & \text{Elsewhere} \end{cases} \tag{II-16}$$

In general, for the cases of interest, the solution will be more complicated.

Now, let us assume that the designer is unaware of or chooses to ignore the interfering target. He would then use a conventional matched filter which is now nonoptimum. One can verify easily that

$$d_c^2 = \frac{2E}{N_o} \left\{ \frac{1}{1 + \gamma_{di}^2(\tau) \frac{2E}{N_o} \sigma_a^2} \right\} \tag{II-17}$$

It is convenient to compute a quantity, "the degradation due to interference." Taking the log of Equation II-15, we have:

$$\log d_o^2 = \log \frac{2E}{N_o} + \log \left[1 - \gamma_{di}^2 (\tau) \frac{\frac{2E}{N_o} \sigma_a^2}{1 + \frac{2E}{N_o} \sigma_a^2} \right] \qquad \text{(II-18)}$$

Since the first term is caused by white noise, the degradation due to interference is just the magnitude of the second term.

Similarly, from Equation II-17,

$$\log d_c^2 = \log \frac{2E}{N_o} - \log \left[1 + \gamma_{di}^2 (\tau) \cdot \frac{2E}{N_o} \sigma_a^2 \right] \qquad \text{(II-19)}$$

The degradation for the conventional and optimum receiver cases is shown in Figure 5.

We observe that in both the good performance region ($\gamma^2 \to 0$) and the bad performance region ($\gamma^2 \to 1$), the optimum receiver is not much better than the conventional receiver. We can see that this result is intuitively logical by looking at Equation II-8.

As $\gamma^2 \to 0$, the coefficient of the second term approaches zero. Physically, $\gamma^2 = 0$ means that the desired signal and the interfering signal are orthogonal. Thus, the interfering signal causes no output in the correlation detector. Clearly, a signal that causes no output cannot affect the performance, and there is no reason to modify the detector.

At the other extreme, as $\gamma^2 \to 1$, the modifying term, $S_d(t-\tau)$, looks more and more like the original term. In the limit, $\gamma^2 = 1$, $S_d(t-\tau) = S_d(t)$ and no modification is necessary.

This simple example illustrates many of the important features of the actual reverberation problem. We may summarize these briefly:

1. The optimum detector tries to subtract out the interfering signal. Since it does not know the signal, it uses the received waveform to estimate the interfering signal and then subtracts out this estimate.

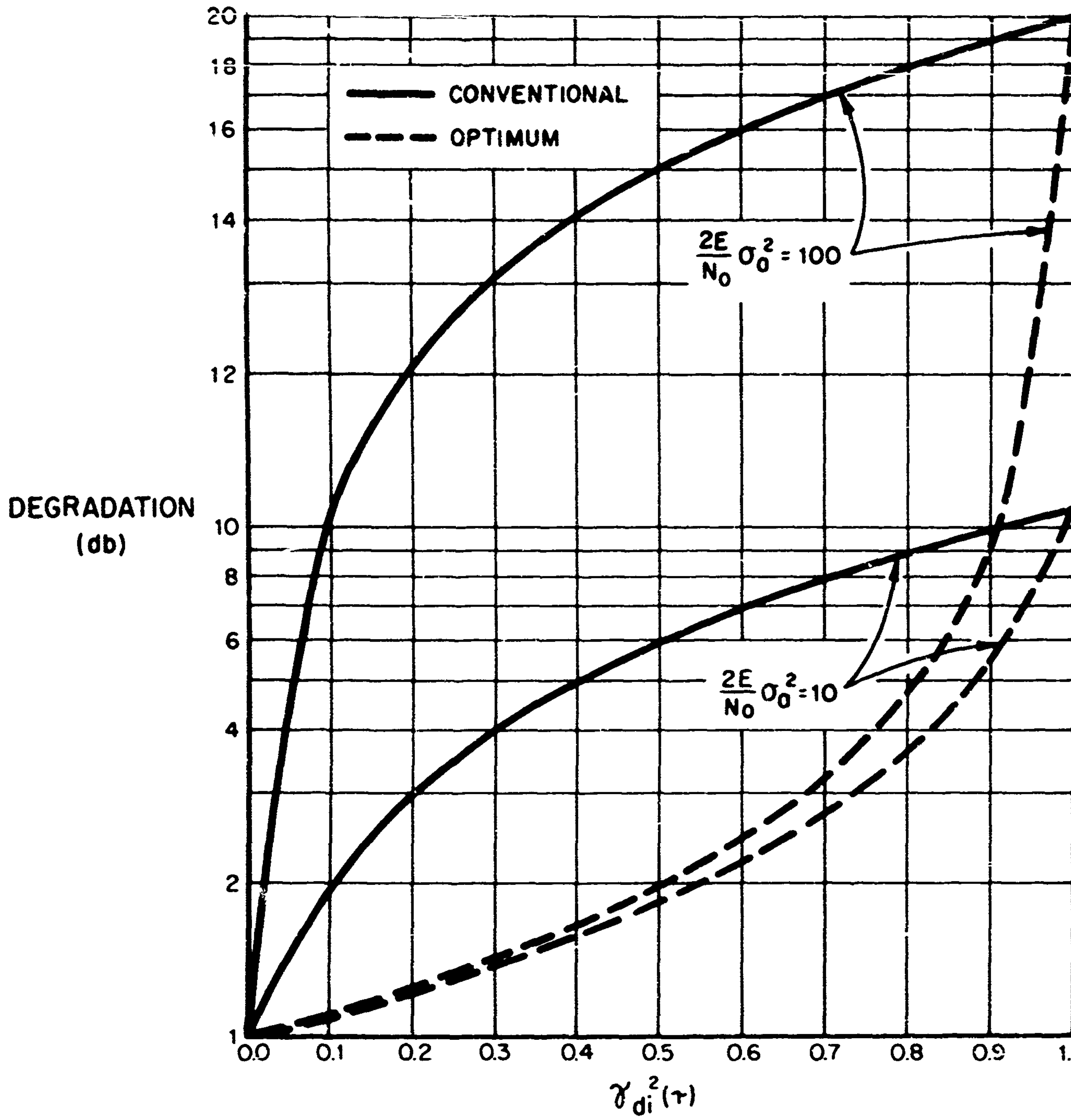

FIGURE 5 DEGRADATION DUE TO INTERFERENCE

2. The choice of signals affects the performance appreciably. The goal of the signal designer is simple to state: make the signal returned from the desired target orthogonal to all of the interference. We will see that in most realistic situations, as we would expect, this is difficult to do.

3. To design an optimum detector, requires some knowledge of the interfering signal. In this simple case, we knew the shape and the probability density of the amplitude. We would expect that as knowledge of the interference decreased, the improvement gained would decrease. In many cases, because of a lack of knowledge or in order to simplify the resulting equipment, one uses a "conventional" receiver. We observed that for certain parameter ranges there was not too much difference between the conventional and the optimum receiver. We will find that in many cases of interest proper signal design is much more important than the difference between a conventional and an optimum receiver.

B. PROBLEM FORMULATION

There are important differences between the simple problem discussed above and the reverberation problem:

1. Instead of one interfering target, there is a large number of interfering returns from reflecting objects.

2. The returned signal from the target is a band-pass waveform. It has a random phase angle which must be taken into account.

We shall see that these differences take us from a tutorial exercise to a reasonably good model of an active sonar in a reverberation environment. The cost of this transition is a great increase in the complexity of the calculations. It is important to emphasize that the concepts in an actual sonar problem are identical to those in the preceding example.

Our model of the reverberation problem is shown in Figure 6.

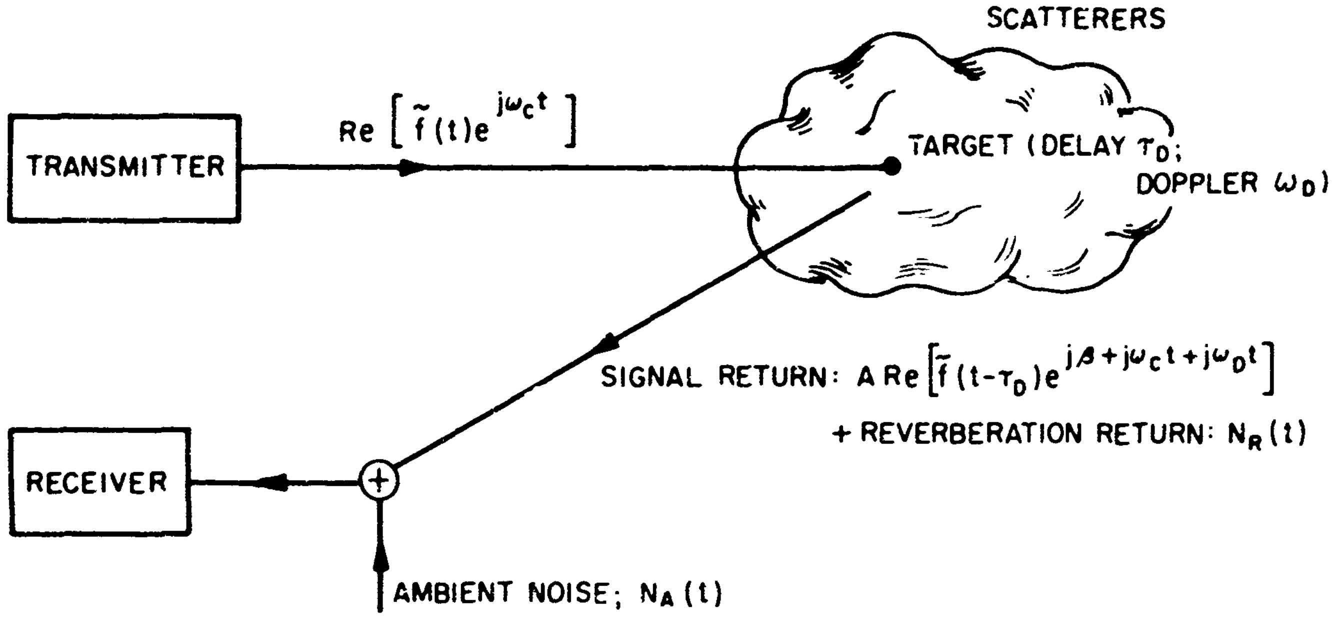

FIGURE 6 CHANNEL MODEL

The transmitted signal is:

$$S_T(t) = \mathrm{Re}\left[f(t)e^{j\omega_c t}\right] = \mathrm{Re}\left[u(t)e^{j\phi(t)}e^{j\omega_c t}\right] = u(t)\cos(\omega_c t + \phi(t) \quad \text{(II-20)}$$

where:

$f(t)$ is the complex envelope/

$u(t)$ is the actual envelope

$\phi(t)$ is the phase

/ For a discussion of the complex representation, see Woodward (Reference 1) and Helstrom (Reference 2).

The returned signal $R(t) = \mathrm{Re}\left[n_r(t)e^{j\omega_c t}\right]$ consists of the following three parts:

1. The reverberation return due to the collection of nonstationary scatterers:

$$N_R(t) = \mathrm{Re}\left[n_r(t)e^{j\omega_c t}\right] \qquad \text{(II-21)}$$

2. The return due to additive Gaussian noise:

$$N_a(t) = \mathrm{Re}\left[n_a(t)e^{j\omega_c t}\right] \qquad \text{(II-22)}$$

3. The return due to a target. This is an attenuated and phase-shifted replica of the transmitted signal.

$$A\,\mathrm{Re}\left[f(t) - \tau_d)e^{+j\theta + j\omega_c t + j\omega_D t}\right] \qquad \text{(II-23)}$$

where θ is uniform $[0, 2\pi]$ and unknown. The first two parts of the return are always present. The third part is only present if a target is present.

As before, we are concerned with deciding whether or not a target is present. In other words, we want to decide between one of two hypotheses:

H_O (no target): $\quad r_o(t) = n_r(t) + n_a(t)$

H_1 (target present): $\quad r(t) = f\,(t - \tau_D)e^{j\omega_D t + j\theta} + n_r(t) + n_a(t)$

If $n_r(t)$ is a complex Gaussian process, the solution to the problem can be expressed in terms of an integral equation. Assuming the integral equation can be solved, one can then construct the optimum receiver.

III. MODEL FOR NON-UNIFORM SCATTERERS

In this section, we develop a model for the reverberation return.

Our approach is a generalization of that in Reference 3 (more readily available References are 4 and 5).

The transmitted signal is:

$$S_T(t) = \mathrm{Re}\left[\tilde{f}(t)\, e^{j\omega_c t}\right] \tag{III-1}$$

We are concerned with $\tilde{f}$ (t) the complex envelope.

$$\tilde{f}(t) = u(t)\, e^{j\phi(t)} \tag{III-2}$$

The complex envelope returned from an individual scatterer (the n^{th} scatterer) is:

$$\tilde{S}_n(t) = Z_n e^{j\omega_n t}\, \tilde{f}(t - t_n) \tag{III-3}$$

where:

Z_n is a complex number which is the magnitude and phase of the echo (i.e., "strength" of echo),

ω_n is the Doppler shift due to the radial velocity of the n^{th} scatterer, and

t_n is the delay due to the position of the n^{th} scatterer.

The entire complex envelope due to reverberation is:

$$\tilde{n}_r(t) = \sum_{\text{all scatterers}} Z_n e^{j\omega_n t}\, \tilde{f}(t - t_n) \tag{III-4}$$

We make the following assumptions regarding the scatterers:

Assumption 1

The distribution along the path obeys a non-homogeneous Poisson law. (See Reference 6.)

The probability that a scatterer exists in the time interval. It is:

$$\Pr\left[1 \text{ event},\ t_\alpha < t < t_\alpha\right] = a\,(t_\alpha)\,dt \qquad \text{(III-5)}$$

All of the properties of a stationary or homogeneous process can be extended easily to a non-homogeneous process. The two properties that we will use are:

Property I: $\Pr(n \text{ events in interval } [-T,T]) = \dfrac{\left(\int_{-T}^{T} a(x)dx\right)^n \exp - \int_{-T}^{T} a(x)dx}{n!}$

(III-6)

Property II: Given that n events occur in the interval $[-T,T]$, the joint probability density of their occurrence times is given by the expression

$$p_{t_1, t_2, \ldots, t_n}\Big|_{n \text{ events in } (-T,T)}[t_1, t_2, \ldots, t_n] =$$

$$\frac{1}{\left[\int_{-T}^{T} a(x)\,dx\right]^n} \cdot a(t_1)\,a(t_2), \ldots, a(t_n) \qquad \text{(III-7)}$$

Assumption 2

The velocity of each scatterer is a random variable. Velocities of different scatterers are independent random variables.

The probability density governing the scatterer velocity is time dependent. The probability that a scatterer occurs in the interval $[t_\alpha, t_\alpha + dt_\alpha]$ and has a velocity (frequency shift)in the range $[\omega_1, \omega_1 + d\omega_1]$ can be described by a joint density,

$$p_{t,\omega}(t_\alpha, \omega_1) = p_{\omega|t}(\omega_1 | t_\alpha)\,p_t(t_\alpha) \equiv p_{\omega|t}(\omega_1 : t_\alpha) \cdot \frac{a(t_\alpha)}{\int_{-T}^{T} a(y)\,dx} \qquad \text{(III-8)}$$

Assumption 3

The strength of each scatterer is a random variable. Strengths of different scatterers are independent random variables. The strength is independent of both position and velocity. We denote this probability density by:

$$p_{Z_1}(Z) = p_{Z_2}(Z) = \ldots p_{Z_n}(Z) \tag{III-9}$$

Using these three assumptions, we want to find the correlation function of the reverberation return. For simplicity, we will assume $E[Z_n] = 0$.≠

The complex envelope of the returned reverberation signal is:

$$\tilde{n}_r(t) = \sum_{m=0}^{\infty} Z_m e^{j\omega_m t} \tilde{f}(t - t_m) \tag{III-10}$$

Then, the correlation function is:

$$\tilde{R}(t_\alpha, t_\beta) = \tfrac{1}{2} < \tilde{r}(t_\alpha) \tilde{r}^*(t_\beta) > \tag{III-11}$$

An easy way to find this is to assume there were n scatterers in the interval (-T, T). If we denote the conditional correlation function based on this assumption by $\tilde{R}_n(t_\alpha, t_\beta)$, then:

$$\tilde{R}(t_\alpha, t_\beta) = \sum_{n=0}^{\infty} \tilde{R}_n(t_\alpha, t_\beta) \cdot \Pr[n \text{ events in } (-T, T)] \tag{III-12}$$

Using Equation III-6, this reduces to:

$$\tilde{R}(t_\alpha, t_\beta) = \sum_{n=0}^{\infty} \tilde{R}_n(t_\alpha, t_\beta) \cdot \frac{\left[\int_{-T}^{T} a(x)\,dx\right]^n \exp - \int_{-T}^{T} a(x)\,dx}{n!} \tag{III-13}$$

≠ We use the symbol E to denote the expectation of a random variable.

Now we find $\widetilde{R}_n(t_\alpha, t_\beta)$[/].

Using Equation III-7 through III-11, we may write:

$$\widetilde{R}_n(t_\alpha, t_\beta) = \frac{1}{2}\frac{1}{\left[\int_{-T}^{T} a(x)\,dx\right]^n} \int_{-T}^{T}\cdots\int_{-T}^{T} dt_1\ldots dt_n\, a(t_1)\ldots a(t_n)$$

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} p_{\omega_1|t_1}(\omega_1 : t_1)\ldots p_{\omega_n|t_n}(\omega_n : t_n)\,d\omega_1\ldots d\omega_n$$

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} p_{Z_1}(Z_1)\ldots p_{Z_n}(Z_n)\,dZ_1\ldots dZ_n$$

$$\sum_{i=1}^{n}\sum_{k=1}^{n} Z_i Z_k^* \,\widetilde{f}(t_\alpha - t_i)\,\widetilde{f}^*(t_\beta - t_k)\, e^{+j\omega_i(t_\alpha - t_i)}\, e^{-j\omega_k(t_\beta - t_k)} \qquad \text{(III-14)}$$

Several observations simplify this expression:

1. $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{Z_i}(Z_i)\, p_{Z_k}(Z_k)\, Z_i Z_k^* = \begin{cases} E|Z_i|^2 & i = k \\ 0 & i \neq k \end{cases} \qquad \text{(III-15)}$$

This reduces the double summation to a single summation.

2. For i = k, the exponential terms reduce to:

$$e^{+j\omega_i(t_\alpha - t_\beta)} \equiv e^{+j\omega_i \tau} \qquad \text{(III-16)}$$

(where $\tau = t_\alpha - t_\beta$)

[/] This approach is similar to Sec. 7-4 in Reference 7. This section follows the original work of Rice (Reference 8).

Now consider the term in the series for $i = q$. After performing the integration with respect to Z_q, the multiple integral becomes:

$$\frac{1}{\left[\int_{-T}^{T} a(x)\,dx\right]^n} \int_{-T}^{T}\cdots\int_{-T}^{T} dt_1 \ldots dt_{q-1}\,dt_{q+1}\ldots dt_n\, a(t_1)\ldots a(t_{q-1})\,a(t_{q+1})\ldots a(t_n)$$

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} p_{\omega_1|t_1}(\omega_1:t_1)\ldots p_{\omega_{q-1}|t_{q-1}}(\omega_{q-1}:t_{q-1})\,p_{\omega_{q+1}|t_{q+1}}(\omega_{q+1}:t_{q+1})$$

$$\ldots p_{\omega_n|t_n}(\omega_n:t_n)\,d\omega_1\ldots d\omega_{q-1}\,d\omega_{q+1}\ldots d\omega_n$$

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} p_{Z_1}(Z_1)\ldots p_{Z_{q-1}}(Z_{q-1})\,p_{Z_{q+1}}(Z_{q+1})\ldots p_{Z_n}(Z_n)\,dZ_1\ldots d_{Z_{q-1}}\,dZ_{q+1}\ldots dZ$$

$$\int_{-T}^{T} a(t_q)dt_q \int_{-\infty}^{\infty} p_{\omega_q|t_q}(\omega_q|t_q)\,d\omega_q \cdot E\left\{|Z_q|^2\right\}\tilde{f}(t_\alpha - t_q)\,\tilde{f}^*(t_\beta - t_q)\,e^{+j\omega_q\tau} \qquad \text{(III-17)}$$

The integrals with respect to variables other than t_q and ω_q are straightforward.

We observe that the integral with respect to ω_q is in the form of a conditional characteristic function:

$$M_{\omega_q:t_q}(\tau:t_q) \equiv \int_{-\infty}^{\infty} p_{\omega_q|t_q}(\omega_q|t_q)\,e^{+j\omega_q\tau}\,d\omega_q \qquad \text{(III-18)}$$

Assuming that T is large, we can neglect end effects.

Since each term in the sum is the same, we have:

$$\tilde{R}_n(t_\alpha, t_\beta) = \frac{n\,E\left[|Z|^2\right]}{\int_{-T}^{T} a(x)\,dx} \int_{-T}^{T} a(t_q)\,M_{\omega_q:t_q}(t_\alpha - t_\beta : t_q)\,\tilde{f}(t_\alpha - t_q)\,\tilde{f}^*(t_\beta - t_q) \qquad \text{(III-19)}$$

Denote the integral by $I(t_\alpha, t_\beta)$. Substituting Equation III-15 into Equation III-9, we have:

$$\widetilde{R}(t_\alpha, t_\beta) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{\left[\int_{-T}^{T} a(x)dx\right]^n \exp - \int_{-T}^{T} a(x)dx}{n!} \cdot \frac{n\,E\left[|Z|^2\right] \cdot I(t_\alpha, t_\beta)}{\int_{-T}^{T} a(x)\,dx} \quad \text{(III-20)}$$

This reduces to:

$$\widetilde{R}(t_\alpha, t_\beta) = \frac{1}{2} E\left[|Z|^2\right] I(t_\alpha, t_\beta) \exp - \int_{-T}^{T} a(x)dx \sum_{n=1}^{\infty} \frac{\left[\int_{-T}^{T} a(x)dx\right]^{n-1}}{(n-1)!} \quad \text{(III-21)}$$

We observe that the sum cancels the exponential term preceding it. Equation III-21 becomes:

$$\widetilde{R}(t_\alpha, t_\beta) = \frac{1}{2} E\left[|Z|^2\right] I(t_\alpha, t_\beta) \quad \text{(III-22)}$$

where

$$I(t_\alpha, t_\beta) = \int_{-T}^{T} a(x)\, M_{\omega_q:x}(t_\alpha - t_{\beta:x})\, \widetilde{f}(t_\alpha - x)\, \widetilde{f}^*(t_\beta - x)\, dx \quad \text{(III-23)}$$

Equation III-23 is valid for the case defined by the original assumptions, and it can be written in several different ways. The two-dimensional correlation function of a signal is defined to be:†

$$\theta_1(\tau, \omega) = \int_{-\infty}^{\infty} \widetilde{f}\left(t - \frac{\tau}{2}\right) \widetilde{f}^*\left(t + \frac{\tau}{2}\right) e^{-j\omega t} dt \quad \text{(III-24)}$$

or

$$\widetilde{f}\left(t - \frac{\tau}{2}\right) \widetilde{f}^*\left(t + \frac{\tau}{2}\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \theta_1(\tau, \omega)\, e^{+j\omega t} d\omega \quad \text{(III-25)}$$

By letting

$t = \dfrac{t_\alpha + t_\beta}{2} - x$, $\tau = t_\beta - t_\alpha$, Equation III-25 becomes:

---

† See, e.g., Woodward (Reference 1) or Siebert (Reference 9).

$$\tilde{f}(t_\alpha - x)\,\tilde{f}^*(t_\beta - x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\theta_1(t_\beta - t_\alpha;\omega)\, e^{+j\omega\left(\frac{t_\alpha + t_\beta}{2} - x\right)}\, d\omega \qquad \text{(III-26)}$$

Substituting Equation III-26 into Equation III-22, we have:

$$\tilde{R}(t_\alpha, t_\beta) = \tfrac{1}{2}\,\frac{E\{|Z|^2\}}{2\pi}\iint a(x)\, M_{\omega_q|x}(t_\alpha - t_\beta{:}x)\,\theta_1(t_\beta - t_\alpha;\omega)\exp +j\omega\left(\frac{t_\alpha + t_\beta}{2} - x\right) d\omega \qquad \text{(III-27)}$$

$$\text{But} \int a(x)\, M_{\omega_q:x}(t_\alpha - t_\beta{:}x)\, e^{-j\omega x}\, dx \equiv S(-\omega;\, t_\alpha - t_\beta) \qquad \text{(III-28)}$$

where

$$S(v_1, v_2) = \iint_{-\infty}^{\infty} e^{+jv_1x_1 + jv_2x_2}\, p_{\tau,\omega}(x_1, x_2)\, dx_1\, dx_2$$

[the characteristic fctn. of the joint density] (III-29)

$\therefore$

$$\tilde{R}_{n_v}(t_\alpha, t_\beta) = \tfrac{1}{2}\,\frac{E\{|Z|^2\}}{2\pi}\int_{-\infty}^{\infty} S(-\omega;\, t_\alpha - t_\beta)\,\theta_1(t_\beta - t_\alpha;\omega)\exp +j\omega\left(\frac{t_\alpha + t_\beta}{2}\right) d\omega \qquad \text{(III-30)}$$

The total interfering noise is the sum of the reverberation noise and the ambient noise.

The total correlation function is:†

$$\tilde{R}_r(t_\alpha, t_\beta) = N_o\,\delta(t_\alpha - t_\beta) + \tilde{R}_{n_r}(t_\alpha, t_\beta) \qquad \text{(III-31)}$$

For a large number of scatterers, one can show that the interfering noise approaches a non-stationary Gaussian process.††

† We assume the ambient noise is a real white Gaussian process with double-sided spectral height $N_0/2$.

†† The technique is similar to that of Reference 4.

The simplest case is when the distribution in range of the scatterers is uniform and the velocity density is uniform.

Let the

$$a(x) = \gamma \qquad -T < x < T \tag{III-32}$$

Physically, γ is the number of scatterers per unit interval (time). Now let $T \to \infty$.

Then, from Equations III-22 and III-23

$$\widetilde{R}(t_\alpha, t_\beta) = \tfrac{1}{2}\gamma E\left[|Z|^2\right] M_{\omega_q}(t_\alpha - t_\beta) \int_{-\infty}^{+\infty} \widetilde{f}(t_\alpha - x)\, \widetilde{f}^*(t_\beta - x)\, dx \tag{III-33}$$

or

letting $u = t_\alpha - x$

$\tau = t_\alpha - t_\beta$

we have:

$$\widetilde{R}(\tau) = \tfrac{1}{2}\gamma E\left[|Z|^2\right] M_{\omega_q}(\tau) \int_{-\infty}^{+\infty} \widetilde{f}(u)\, \widetilde{f}^*(u+\tau)\, du \tag{III-34}$$

But the integral is just $\widetilde{R}_f(\tau)$

$$\widetilde{R}_f(\tau) \equiv \int_{-\infty}^{\infty} \widetilde{f}(u)\, \widetilde{f}^*(u+\tau)\, du \tag{III-35}$$

Then,

$$\widetilde{R}_{n_r}(\tau) = \tfrac{1}{2}\gamma E\left[|Z|^2\right] M_{\omega_q}(\tau)\, \widetilde{R}_f(\tau) \equiv \tfrac{1}{2} I_{av} M_{\omega_q}(\tau)\, \widetilde{R}_f(\tau) \tag{III-36}$$

Our principal results in this section are Equation III-30, the correlation function for the reverberation return for the case of non-uniform distribution in range of the scatterers, and Equation III-36, the correlation function for the reverberation return for the case of uniform distribution in range of the scatterers. Since the process is Gaussian it is completely characterized by its correlation function. In the next section, we use this to develop the optimum receiver structure.

IV. DERIVATION OF OPTIMUM RECEIVER STRUCTURE

In this section, we derive some general results regarding receiver structure and performance that we will need for our specific problem.

A. STRUCTURE/

We first derive the structure of the optimum receiver and the conventional receiver. Then, we find expressions for their performance.

As pointed out in the introduction, the optimum receiver solves the hypothesis testing problem. One can demonstrate that for many interesting criteria (e.g., Bayes, Neyman - Pearson, Minimax) the solution reduces to one of forming the likelihood ratio and comparing it with a threshold./ The value of the threshold will depend on the decision criterion and relative costs. We will not concern ourselves with choosing a specific value of the threshold but only with forming the test statistic.

We are concerned with detecting a signal which is a member of the ensemble $S(t, \beta)$, where:

$$S(t, \beta) = \mathrm{Re}\left[\tilde{S}_d(t) e^{j\beta} e^{j\omega_c t}\right] \tag{IV-1}$$

where $$p_\beta(\beta) = \frac{1}{2\pi} \qquad 0 < \beta < 2\pi \tag{IV-2}$$

and $$\tilde{S}_d(t) = Af(t - \tau_d)\, e^{j\omega_d t} \tag{IV-3}$$

To form the likelihood ratio, we expand the complex envelope $r(t)$ using a Karhunen-Loeve expansion.//

/ Helstrom (Reference 2), Chapter 3.

// See Reference 7, Chapter 6.

The orthonormal functions of this expansion are the eigenfunctions of the integral equation:

$$\lambda_n \phi_n(t_\alpha) = \int_{-T}^{T} \tilde{R}_{r_o}(t_\alpha, t_\beta)\, \phi_n(t_\beta)\, dt_\beta \tag{IV-4}$$

where $\tilde{R}_{r_o}(t_\alpha, t_\beta)$ is the complex covariance of $r_o(t)$.

$$\tilde{R}_{r_o}(t_\alpha, t_\beta) = \frac{1}{2} E\left[\tilde{r}_o(t_\alpha)\, \tilde{r}_o^*(t_\beta)\right] \tag{IV-5}$$

Then,

$$\tilde{r}(t) = \sum \tilde{r}_n \phi_n(t) \tag{IV-6}$$

$$\tilde{r}_n = \int_{-T}^{T} \tilde{r}(t)\, \phi_n^*(t)\, dt = x_n + j\, y_n \tag{IV-7}$$

We now want to find the statistics of x_n and y_n under the two hypotheses.

Under the no-target condition (hypothesis H_0), the complex envelope of the received signal is

$$r_o(t) = n_r(t) + n_a(t) \tag{IV-8}$$

Under the target present condition (hypothesis H_1), it is

$$r_1(t) = S_d(t)e^{j\beta} + n_r(t) + n_a(t) \tag{IV-9}$$

Clearly, x_n and y_n are Gaussian random variables under either hypothesis.

Under hypothesis H_0,

$$E\left[x_n\right] = E\left[y_n\right] = 0 \tag{IV-10}$$

while under hypothesis H_1,

$$E\left[x_n + j\, y_n\right] = E\left[r_n\right] = E\left[\int_{-T}^{T} \tilde{S}_d(t)\, e^{j\beta} \cdot \phi_n(t)\, d\right] = S_n e^{j\beta} \qquad \text{(IV-11)}$$

where

$$S_n = \left[\int_{-T}^{T} \tilde{S}_d(t)\, \phi_n(t)\, dt\right] \qquad \text{(IV-12)}$$

The covariances are independent of the hypothesis and are:

$$E\left[(r_n - \bar{r}_n)(r_m^* - \bar{r}_m^*)\right] E = \int_{-T}^{T} du \int_{-T}^{T} dv\ E\left[\tilde{r}(u)\, \tilde{r}^*(v)\right] \phi_n^*(u)\, \phi_m(v)$$

$$= 2 \int_{-T}^{T} du \int_{-T}^{T} dv\ \tilde{R}_{r_o}(u, v)\, \phi_n^*(u)\, \phi_m(v) \qquad \text{(IV-15)}$$

Using Equation IV-4, we have:

$$E\left[(r_n - \bar{r}_n)(r_m^* - \bar{r}_m^*)\right] = 2\lambda_m \int_{-T}^{T} du\ \phi_n^*(u)\, \phi_m(u) = 2\lambda_m \delta_{nm} \qquad \text{(IV-16)}$$

Similarly, using the properties of the expansion, we have:

$$E\left[r_n r_m\right] = 0 \qquad \text{(IV-17)}$$

Since $r_n = x_n + j\, y_n$ (IV-18)

we have $E\left[(x_n - \bar{x}_n)(x_m - \bar{x}^*_m)\right] = \lambda_n \delta_{nm}$; $E\left[(y_n - \bar{y}_n)(y_m - \bar{y}^*_m)\right] = \lambda_n \delta_{nm}$ (IV-19)

and $$E\left[(x_n - \bar{x}_n)(y_m - \bar{y}^*_m)\right] = 0 \qquad \text{(IV-20)}$$

Therefore, the x_n and y_n are statistically independent random variables. Now write the likelihood ratio:

$$\Lambda = \frac{p(\tilde{r} \mid H_1)}{p(\tilde{r} \mid H_0)} \qquad \text{(IV-21)}$$

Now

$$p(\tilde{r} \mid H_1) = \int_0^{2\pi} p(\beta)\, d\beta \left[\prod_{i=1}^{K} \frac{1}{2\pi\lambda_i}\right] \exp - \sum_{n=1}^{K} \frac{(x_n - E(x_n))^2 + (y_n - E(y_n))^2}{2\lambda_n} \qquad \text{(IV-22)}$$

or

$$p(\tilde{r} \mid H_1) = \int_0^{2\pi} p(\beta)\, d\beta \left[\prod_{i=1}^{K} \frac{1}{2\pi\lambda_i}\right] \exp - \sum_{n=1}^{K} \frac{\left|r_n - s_n e^{j\beta}\right|^2}{2\lambda_n} \qquad \text{(IV-23)}$$

and

$$p(\tilde{r} \mid H_0) = \left[\prod_{i=1}^{K} \frac{1}{2\pi\lambda_i}\right] \exp - \sum_{n=1}^{K} \frac{\left|r_n\right|^2}{2\lambda_n} \qquad \text{(IV-24)}$$

We have:

$$\Lambda = \int_0^{2\pi} \frac{1}{2\pi}\, d\beta \exp + \left\{ \sum_{n=1}^{K} \frac{2\mathrm{Re}\left[s^*_n r_n e^{-j\beta}\right] - \left|s_n\right|^2}{2\lambda_n} \right\} \qquad \text{(IV-25)}$$

Now define

$$A_K e^{j\alpha} = \sum_{n=1}^{K} \frac{s_n^* r_n}{\lambda_n} \tag{IV-26}$$

where A_K is real and positive.

Then, we may write:

$$\Lambda = \left[\frac{1}{2\pi} \int_0^{2\pi} d\beta \; \exp - \left\{ A_K \cos(\alpha - \beta) \right\} \right] \exp - \sum_{n=1}^{K} \frac{|s_n|^2}{2\lambda_n} \tag{IV-27}$$

But the expression in the bracket is just $I_0(A_K)$ where I_0 is a modified Bessel function of the first kind and order zero. That is

$$\Lambda = I_0(A_K) \exp - \sum \frac{|s_n|^2}{2\lambda_n}$$

Now $I_0(\cdot)$ is monotone for positive and negative arguments and symmetric around the origin. Letting $K \to \infty$, we see that an adequate statistic is:

$$\lim_{K\to\infty} \left| \sum_{n=1}^{K} \frac{s_n^* v_n}{\lambda_n} \right| = \left| \int_{-T}^{T} \tilde{q}^*(t)\, \tilde{r}(t)\, dt \right| \tag{IV-28}$$

where $q^*(t)$ is the solution to the integral equation.

$$\tilde{S}_d(t_\alpha) = \int_{-T}^{T} \tilde{R}_{r_o}(t_\alpha, t_\beta)\, \tilde{q}(t_\beta)\, dt_\beta \tag{IV-29}$$

The second term in Equation IV-27 is incorporated into the threshold.

The desired optimum operations are shown in Figure 7. The desired operations can be realized physically by passing $r(t)$ through a narrow-band filter whose complex impulse response is

$$\tilde{h}_{opt}(\tau) = \tilde{q}^{*}(T-\tau) \qquad -T < \tau < T$$

$$\tilde{h}_{opt}(\tau) = 0 \qquad \text{elsewhere} \tag{IV-30}$$

and detecting the output envelope. This realization is shown in Figure 8.

We now want to consider three cases:

1. The scatterers are uniform in range. This leads to a stationary process for $n_T(t)$ (see Equation IV-36). One solves Equation IV-29 to find the optimum receiver.

2. The scatterers are non-uniform in range. This leads to a non-stationary process for $n_T(t)$. One solves Equation IV-29 to find the optimum receiver.

3. The reverberation return is ignored in finding the optimum receiver. We assume:

$$\tilde{R}_{r_o}(t_\alpha, t_\beta) = N_o \delta(t_\alpha - t_\beta) \tag{IV-31}$$

Then,

$$\tilde{q}(t_\alpha) = \frac{1}{N_o} \tilde{S}_d(t_\alpha) \tag{IV-32}$$

This is called a conventional receiver. We then investigate the performance of the conventional receiver for non-uniform and uniform scatterer distributions.

In the next section, we consider cases (1) and (3).

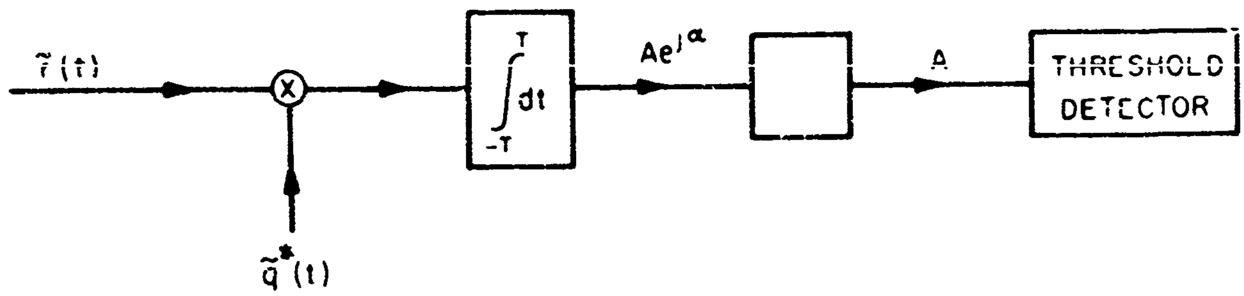

FIGURE 7 DESIRED OPERATIONS (COMPLEX ENVELOPES)

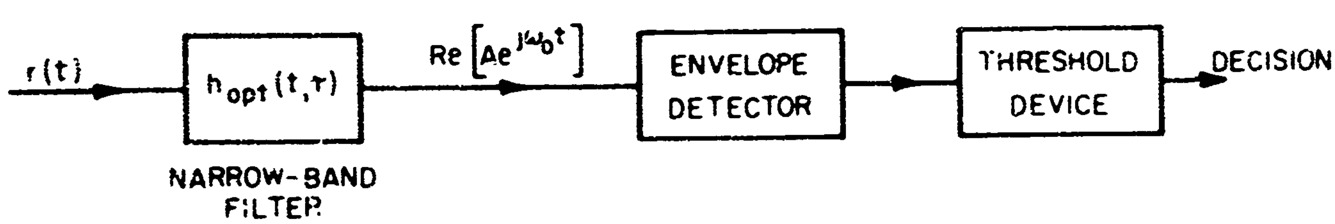

FIGURE 8 PHYSICAL REALIZATION

B. PERFORMANCE OF OPTIMUM RECEIVER/

Our decision is based on comparing the magnitude of A with some threshold Λ_o.

We must consider the statistics of $|A|$ under hypothesis H_0 and hypothesis H_1.

$$\Pr[\text{False Alarm}] = \int_{\Lambda_o}^{\infty} p_o(A)\, dA \doteq P_F \qquad \text{(IV-33)}$$

Now

$$A = | x + jy | \qquad \text{(IV-34)}$$

where

$$x = \mathrm{Re} \int_{-T}^{T} \tilde{q}^*(t)\, r(t)\, dt \qquad \text{(IV-35)}$$

and

$$y = \mathrm{Im} \int_{-T}^{T} \tilde{q}^*(t)\, \tilde{r}_o(t)\, dt \qquad \text{(IV-36)}$$

$$\mathrm{Var}\ x = \mathrm{Var}\ y = \int_{-T}^{T} \tilde{q}^*(t)\, \tilde{s}_d(t)\, dt \triangleq d_o^2 \qquad \text{(IV-37)}$$

and

$$p_o(A) = \frac{A}{d_o^2} e^{-\frac{A^2}{2d_o^2}} \qquad \text{(IV-38)}$$

/ Our derivation is rather sketchy. The details are on pp. 149-156 of Helstrom (Reference 2).

Substituting Equation IV-38 into Equation IV-33 and evaluating, we have:

$$P_F = \exp - \frac{\Lambda_o^2}{2d_o^2} \tag{IV-39}$$

Similarly, one can show, under hypothesis H_1, that

$$\bar{x} = d_o^2 \cos \theta \tag{IV-40}$$

$$\bar{y} = d_o^2 \sin \theta \tag{IV-41}$$

and the probability of detection is:

$$\Pr\left[\text{detection}\right] = Q(d_o, \frac{\Lambda_o}{d_o}) \doteq P_D \tag{IV-42}$$

where Q(a,b) is Marcum's Q function. (See References 8, 11, and 12.)

$$Q(a,b) = \int_b^\infty x \exp - \frac{(x^2+a^2)}{2} I_o(ax)\, dx$$

Observe that from Equations IV-39 and IV-42, we may write:

$$P_D = Q(d_o, -2\ln P_F) \tag{IV-42a}$$

One can plot P_D vs. d_o^2 as a function of P_F. This curve is shown in Figure 9.†

When the correlation function is stationary, there is a simple expression for d_o^2 in terms of the various spectra. For simplicity, assume that the observation interval is infinite.

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† Our figure is similar to Figure V-2, p. 155, Reference 2. An earlier reference is Reference 10.

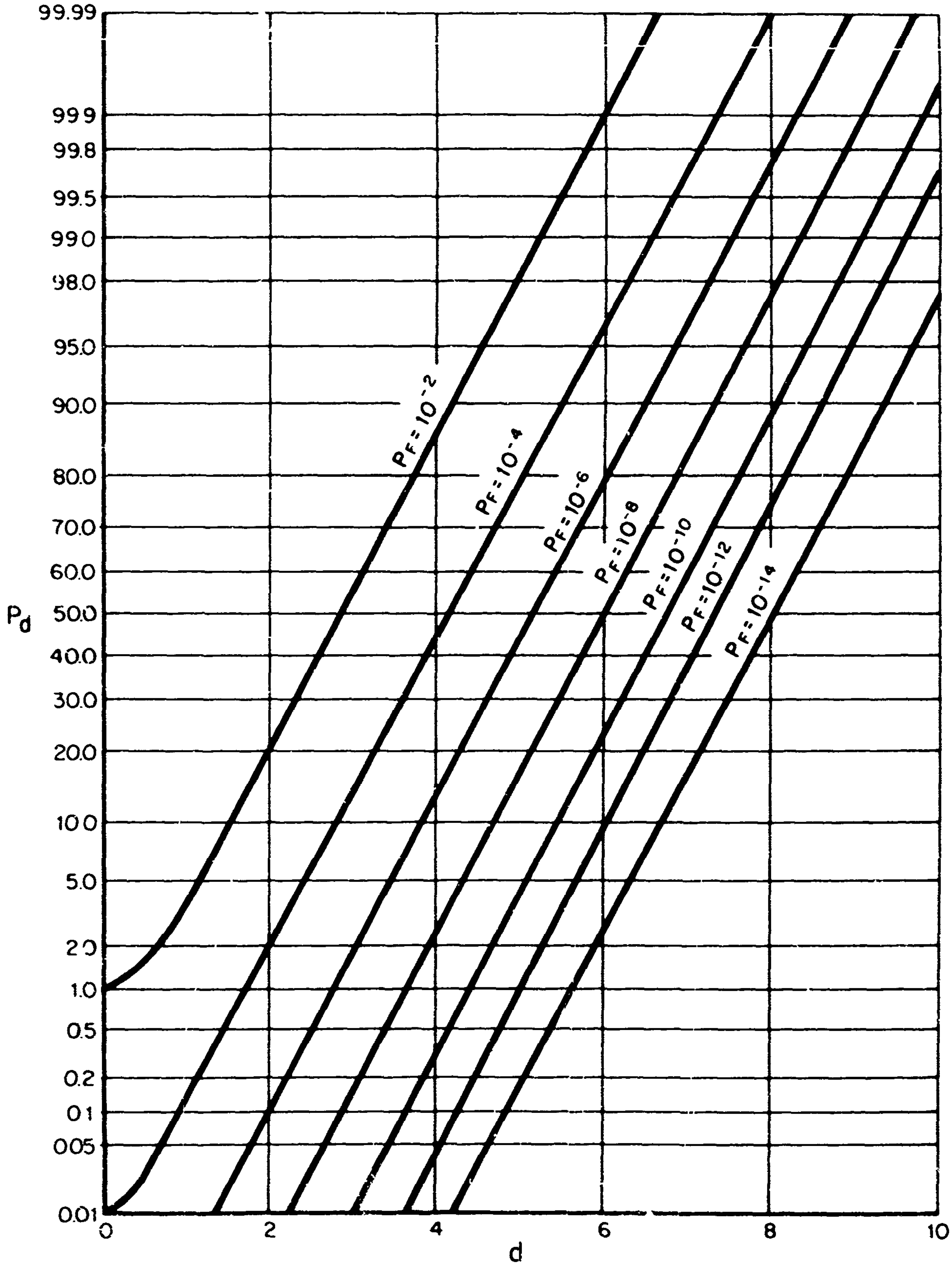

FIGURE 9 RECEIVER OPERATING CHARACTERISTIC: OPTIMUM RECEIVER

Then, Equation IV-29 becomes:

$$\tilde{S}_d(t_\alpha) = \int_{-\infty}^{\infty} \tilde{R}(t_\alpha - t_\beta)\, \tilde{q}(t_\beta)\, dt_\beta \qquad \text{(IV-43)}$$

and Equation IV-37 reduces to:

$$d_o^2 = \int_{-\infty}^{\infty} \tilde{q}^*(t_\alpha)\, \tilde{S}_d(t_\alpha)\, dt_\alpha \qquad \text{(IV-44)}$$

Equation IV-43 can be solved using Fourier transforms. Transforming, we have:

$$\tilde{Q}(\omega) = \frac{S_d(\omega)}{S_R(\omega)} = \frac{S_d(\omega)}{N_o + S_{n_r}(\omega)} \qquad \text{(IV-45)}$$

and using Parseval's theorem:

$$d_o^2 = \int_{-\infty}^{\infty} \frac{\left| S_d(\omega) \right|^2}{N_o + S^*_{n_r}(\omega)} \frac{d\omega}{2\pi} \qquad \text{(IV-46)}$$

In the absence of reverberation, the receiver is specified by Equation IV-32 and

$$d_o^2 = \frac{1}{N_o} \int \tilde{S}_d(t)\, \tilde{S}^*_d(t)\, dt = \frac{2E_r}{N_o} \qquad \text{(IV-47)}$$

Observe that d_o^2 is just the S/N ratio at the receiver output.

C. PERFORMANCE OF CONVENTIONAL RECEIVER

1. General Error Expressions

As we pointed out previously, what we mean by a conventional receiver is one which is designed under the assumption that the interference is additive white Gaussian noise. Clearly, when this assumption is correct, the resulting receiver is optimum. Whenever reverberation is present, this "conventional" receiver will not provide the optimum processing. Because the conventional receiver frequently is far easier to implement than the optimum receiver, we want to find out how far from optimum the conventional receiver is.

In this case,

$$\tilde{q}_c^*(t) = \frac{1}{N_o} \tilde{f}(t) \tag{IV-48}$$

Under hypothesis H_0,

$$x + jy = \int_{-T}^{T} \tilde{q}^*_c(t)\, \tilde{r}_o(t)\, dt = \int_{-T}^{T} \tilde{q}_c^*(t)\, \tilde{n}_T(t)\, dt \tag{IV-49}$$

$$E\left\{ \left| x + jy \right|^2 \right\} = \int_{-T}^{T} du \int_{-T}^{T} dv\, \tilde{q}^*(u)\, \tilde{q}(v)\, E\left[\tilde{r}_o(u)\, \tilde{r}_o^*(v) \right] \tag{IV-50}$$

or

$$\mathrm{Var}[x] + \mathrm{Var}[y] = \frac{2}{N_o^2} \int_{-T}^{T} du \int_{-T}^{T} dv\, \tilde{S}_d^*(u)\, \tilde{S}_d(v)\, \tilde{R}_{r_o}(u, v) = 2\sigma_c^2 \tag{IV-51}$$

Then, in a manner identical to that used to derive Equations IV-33, IV-38, and IV-39, we obtain

$$P_F = \exp - \frac{\Lambda_o^2}{2\sigma_c^2} \tag{IV-52}$$

Under hypothesis H_1,

$$\bar{x} + j\bar{y} = \int_{-T}^{T} \tilde{q}^*(t) \; E\left[r_1(t)\right] dt$$

$$= \int_{-T}^{T} \frac{1}{N_o} \tilde{S}_d^*(t) \; S_d(t) \, e^{j\theta} \, dt \; = \; \frac{2E_r}{N_o} e^{j\theta} \qquad \text{(IV-53)}$$

or

$$\bar{x} = \frac{2E_r}{N_o} \cos\theta \qquad \text{(IV-54)}$$

and

$$\bar{y} = \frac{2E_r}{N_o} \sin\theta \qquad \text{(IV-55)}$$

Then

$$p_{H_1}(x, y \mid \theta) = \frac{1}{2\pi\sigma_c^2} \exp - \left\{ \frac{\left(x - \frac{2E_r}{N_o}\cos\theta\right)^2 + \left(y - \frac{2E_r}{N_o}\sin\theta\right)^2}{2\sigma_c^2} \right\} \qquad \text{(IV-56)}$$

$$p_{H_1}(x, y \mid \theta) = \frac{1}{2\pi\sigma_c^2} \exp - \left\{ \frac{x^2 + y^2 + \left(\frac{2E_r}{N_o}\right)^2 - 2 \cdot \frac{2E_r}{N} (x\cos\theta + y\sin\theta)}{2\sigma_c^2} \right\} \qquad \text{(IV-57)}$$

$$P_d(\theta) = \iint\limits_{R > \Lambda_o} p_{H_1}(x, y | \theta)\, dx\, dy \qquad \text{(IV-58)}$$

Changing to polar coordinates and integrating, we have:

$$P_d(\theta) = \frac{1}{2\pi\sigma_c^2} \int_{\Lambda_o}^{\infty} dz \int_0^{2\pi} d\phi \cdot Z \exp - \left\{ \frac{Z^2 + \left(\frac{2E_r}{N_o}\right)^2 - 2 \cdot \frac{2E_r}{N_o} Z \cos(\phi - \theta)}{2\sigma_c^2} \right\} \qquad \text{(IV-59)}$$

The answer is not a function of θ, so we have:

$$P_d = \frac{1}{\sigma_c^2} \int_{\Lambda_o}^{\infty} Z\, dZ \exp - \left\{ \frac{Z^2 + \left(\frac{2E_r}{N}\right)^2}{2\sigma_c^2} \right\} I_o \left(\frac{2E_r Z}{N\sigma_c^2} \right) \qquad \text{(IV-60)}$$

Just as in the case of the optimum receiver, we must express P_d in terms of the Q function.

Now

$$Q(\alpha, \beta) = \int_{\beta}^{\infty} x\, e^{-\frac{(x^2 + \alpha^2)}{2}} I_o(\alpha x)\, dx \qquad \text{(IV-61)}$$

Letting: $\frac{Z}{\sigma_c} = x$ in Equation IV-60, we obtain:

$$P_d = \int_{\frac{\Lambda_o}{\sigma_c}}^{\infty} x\, dx \exp - \left\{ \frac{x^2 + \left[\frac{2E_r}{N_o \sigma_c}\right]^2}{2} \right\} I_o \left\{ \frac{2E_r}{N\sigma_c} \cdot x \right\} = Q\left(\frac{2E_r}{N_o \sigma_c}, \frac{\Lambda_o}{\sigma_c} \right) \qquad \text{(IV-62)}$$

Just as in the optimum case, it is useful to define a S/N ratio.

$$d_c^2 \triangleq \frac{\left[\text{mean of output} \mid H_1\right]^2}{\text{variance of output}} = \frac{\left[\frac{2E_r}{N_o}\right]^2}{\sigma_c^2} \qquad \text{(IV-63a)}$$

Then, we may write

$$P_d = Q\left(d_c, \frac{\Lambda_o}{\sigma_c}\right) \qquad \text{(IV-63b)}$$

Using Equations IV-52 and IV-63b, we may write:

$$P_d = Q\,(d_c, \; -2\ln P_F) \qquad \text{(IV-63c)}$$

Thus, the quantity d_c completely characterizes the performance of the conventional receiver./ For a given P_F, we may use Figure 9.

We now want to obtain some simpler expressions for d_c^2 for the stationary and the non-stationary cases.

2. Stationary Case

For the stationary, infinite interval case, a simple expression can be obtained in terms of the various spectra.

For the stationary case, Equation IV-51 becomes

$$\sigma_c^2 = \frac{1}{N_o^2} \int du \int dv \; \tilde{S}_d^*(u) \; \tilde{S}_d(v) \; \tilde{R}_T(u-v) \qquad \text{(IV-64)}$$

/ This was pointed out by C. Boardman, M.I.T.

From Equation III-36,

$$\widetilde{R}_{n_r}(u-v) = \frac{\gamma E\left\{|Z|^2\right\}}{2} M_{\omega_q}(u-v)\,\widetilde{R}_f(u-v) = \frac{1}{2} I_{av} M_{\omega_q}(u-v)\,\widetilde{R}_f(u-v) \quad \text{(IV-65)}$$

Defining,

$$S_d(\omega) = \int \widetilde{S}_d(t)\, e^{-j\omega t} dt \quad \text{(IV-66)}$$

and

$$S_r(\omega) = \int \widetilde{R}(t)\, e^{-j\omega t} dt \quad \text{(IV-67)}$$

Then,

$$\sigma_c^2 = \frac{1}{N_o^2} \int\int du\, e^{+ju\omega}\, \widetilde{S}_d^*(u) \int dv\, e^{-jv\omega}\, \widetilde{S}_d(v)\, S_r(\omega)\, \frac{d\omega}{2\pi} \quad \text{(IV-68)}$$

Using Equation IV-66, we obtain:

$$\sigma_c^2 = \frac{1}{N_o^2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left|S_d(\omega)\right|^2 S_r(\omega) \quad \text{(IV-69)}$$

Now, from Equation III-31

$$S_r(\omega) = N_o + S_{n_r}(\omega), \qquad \text{where} \quad S_{n_r}(\omega) = F\left[\widetilde{R}(t_\alpha - t_\beta)\right] \quad \text{(IV-70)}$$

Using Parseval's theorem, we have

$$\sigma_c^2 = \frac{2E_r}{N_o} + \frac{1}{N_o^2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left| S_d(\omega) \right|^2 S_{n_r}(\omega) \tag{IV-71}$$

From Equation IV-63, we observe that

$$d_c^2 = \frac{\dfrac{2E_r}{N_o}}{1 + \dfrac{1}{2E_r N_o} \displaystyle\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left| S_d(\omega) \right|^2 S_{n_r}(\omega)} \tag{IV-72}$$

3. Evaluation of $d^2_{conv.}$ for Non-Stationary Case

From Equations IV-51 and III-31, we observe that:

$$\sigma_c^2 = \frac{2E_r}{N_o} + \frac{1}{N_o^2} \iint \widetilde{S}_d(t_\alpha) \widetilde{S}_d^*(t_\alpha) \widetilde{R}_{n_r}(t_\alpha, t_\beta)\, dt_\alpha\, dt_\beta \equiv \sigma_1^2 + \sigma_{rc}^2 \tag{IV-73}$$

For our particular problem, the desired signal is an attenuated replica of the transmitted envelope $\widetilde{f}(t)$ which has been shifted in frequency and delayed in time.

Let

$$\widetilde{S}_T(t) = \sqrt{2E_t}\ \widetilde{f}(t) \tag{IV-74}$$

and

$$\widetilde{S}_d(t) = \sqrt{2E_r}\ \widetilde{f}(t-\tau_d)\, e^{+j\omega_d t}$$

Then, σ^2_{rc} is:

$$\sigma^2_{rc} = \frac{2E_r}{N_o^2} \iint \tilde{f}^*(t_\alpha - \tau_d)\, \tilde{f}(t_\beta - \tau_d)\, \tilde{R}(t_\alpha, t_\beta)\, e^{-j\omega_d t_\alpha + j\omega_d t_\beta} dt_\alpha dt_\beta \qquad \text{(IV-75)}$$

From Equation III-23:

$$\tilde{R}(t_\alpha, t_\beta) = (2E_t) \frac{E\{|Z|^2\}}{2} \int a(x)\, M_{\omega_q|x}(t_\alpha - t_\beta : x)\, \tilde{f}(t_\alpha - x)\, \tilde{f}^*(t_\beta - x)\, dx \qquad \text{(IV-76)}$$

We recall that

$$M_{\omega_q|x}(t_\alpha - t_\beta : x) = \int_{-\infty}^{+\infty} p_{\omega_q|x}(\omega : x)\, e^{+j\omega(t_\alpha - t_\beta)} d\omega \qquad \text{(IV-77)}$$

We have:

$$\sigma^2_{rc} = \frac{4E_t E_r \frac{1}{2} E\{|Z|^2\}}{2N_o^2} \iiiint dt_\alpha\, dt_\beta\, dx\, d\omega\, \tilde{f}^*(t_\alpha - \tau_d)\, \tilde{f}(t_\beta - \tau_d)\, \tilde{f}(t_\alpha - x)$$

$$\tilde{f}(t_\beta - x)\, a(x)\, p_{\omega_q|x}(\omega : x)$$

$$\exp\left\{ -j\omega_d t_\alpha + j\omega_d t_\beta + j\omega t_\alpha - j\omega t_\beta \right\} \qquad \text{(IV-78)}$$

Re-arranging the terms, we have:

$$\sigma_{rc}^2 = \frac{4E_tE_r \frac{1}{2} E\{|z|^2\}}{2N_o^2} \int dx \int d\nu \; a(x)\, p_{\nu_q|x}(\nu:x)$$

$$\int dt_\alpha \, \tilde{f}(t_\alpha - x)\, \tilde{f}^*(t_\alpha - \tau_d) \exp\left\{-jt_\alpha(\nu_d - \nu)\right\}$$

$$\int dt_\beta \, \tilde{f}^*(t_\beta - x)\, f(t_\beta - \tau_d) \exp\left\{+jt_\beta(\nu_d - \nu)\right\} \qquad \text{(IV-79)}$$

First, observe that the fourth integral is just the complex conjugate of the third integral. Second, observe that the third integral is identical (except for a phase shift) to the two-dimensional correlation function defined in Equation III-24. With these observations, we have:

$$\int dt_\alpha \, \tilde{f}(t_\alpha - x)\, \tilde{f}^*(t_\alpha - \tau_d) \exp\left\{-jt_\alpha(\nu_d - \nu)\right\}$$

$$\int dt_\beta \, \tilde{f}^*(t_\beta - x)\, f(t_\beta - \tau_d) \exp\left\{+jt_\beta(\nu_d - \nu)\right\}$$

$$= \left|\theta(\tau_d - x; \nu_d - \nu)\right|^2 \triangleq \chi(\tau_d - x; \nu_d - \nu) \qquad \text{(IV-80)}$$

Observe that we normalized $\tilde{f}(t)$, so that $\chi(0,0) = 1$.

Equation IV-80 is just a definition of the signal ambiguity function. See, e.g., Woodward (Reference 1) or Siebert (Reference 9).

Next, we observe that the term $a(x)\, p_{\nu_q|x}(\nu:x)$ is just the joint probability density of the scatterers in delay and Doppler.

We will call this the scattering function

$$s(x;\omega) \stackrel{\Delta}{=} a(x)\, p_{\omega_q \mid x}(\omega{:}x) \tag{IV-81}$$

Substituting Equations IV-80 and IV-81 into Equation IV-79, we have:

$$\sigma^2_{rc}(\tau_d;\omega_d) = \frac{4E_tE_r \frac{1}{2}E\{|Z|^2\}}{2N_o^2} \int dx \int d\omega\, s(x{:}\omega)\, \Upsilon(\tau_d - x; \omega_d - \omega) \tag{IV-82}$$

Therefore, for the conventional receiver $\sigma^2_{rc}(\tau_d;\omega_d)$ can be expressed as a two-dimensional convolution of the reverberation scattering function and the signal ambiguity function./

Combining Equations IV-82 and IV-73, we have:

$$\sigma^2_c = \frac{2E_r}{N_o}\left[1 + \frac{\frac{1}{2}E\{|Z|^2\}E_t}{N_o}\iint dx\, d\omega\, s(x{:}\omega)\, \Upsilon(\tau_d - x; \omega_d - \omega)\right] \tag{IV-83}$$

From Equation IV-63, we have:

$$d^2_c = \frac{\dfrac{2E_r}{N_o}}{1 + \dfrac{\frac{1}{2}E\{|Z|^2\}E_t}{N_o}\displaystyle\iint dx\, d\omega\, s(x{:}\omega)\, \Upsilon(\tau_d - x; \omega_d - \omega)} \tag{IV-84}$$

In this section, we have derived the structure of the optimum receiver. We observed that the output S/N ratio, d_o^2, provided a reasonable characterization of the receiver performance. Frequently, the structure of the optimum

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/ This particular form is not new. Westerfield and Stewart (References 14 and 15) obtain a similar relation. Green (Reference 16) describes its application to radar astronomy.

receiver is complex or requires knowledge that might not be available. In this case, one commonly uses a conventional receiver. We observed that the performance depended on two parameters d_c^2 and σ_c^2. To keep the subsequent work from getting immersed in details, we decided to use the single parameter d_c^2 as a basis of comparison. The principal results that we will use in the subsequent work are Equation IV-44 (and its various modified forms; e.g., Equations IV-46 and IV-47) for the optimum receiver and Equations IV-72 and IV-84 for the conventional receiver.

In the next section, we will derive some general properties relating to signal design and processing in the presence of non-white Gaussian noise. After deriving these properties we will return to the reverberation problem of interest.

V. SIGNAL DESIGN PROPERTIES

In the preceding sections, we have formulated the detection problem in the presence of reverberation. The pertinent results can be summarized briefly. The optimum receiver formulated a test statistic

$$S = \int_{-T}^{T} \tilde{q}^*(t)\, \tilde{r}(t)\, dt \qquad \text{(V-1)}$$

where $\tilde{q}(t)$ is the solution to the integral equation.

$$\tilde{S}_d(t_\alpha) = \int_{-T}^{T} \tilde{R}_{n_T}(t_\alpha, t_\beta)\, \tilde{q}(t_\beta)\, dt_\beta \quad -T < t_\alpha < +T \qquad \text{(V-2)}$$

The performance of the system depends on a quantity d_o^2, where

$$d_o^2 = \int_{-T}^{T} \tilde{q}^*(t)\, S_d(t)\, dt \qquad \text{(V-3)}$$

When the interference has a "white" spectrum, i.e.,

$$\tilde{R}_{n_T}(t_\alpha, t_\beta) = N_o u_o(t_\alpha - t_\beta) \qquad \text{(V-4)}$$

then

$$\tilde{q}(t_\beta) = \frac{1}{N_o} \tilde{S}_d(t_\alpha) \qquad \text{(V-5)}$$

and

$$d_o^2 = \frac{2E_r}{N_o} \qquad \text{(V-6)}$$

The performance depends only on the received energy and not on the signal shape.

For any other spectrum, d_o^2 depends on the signal shape. In this section, we derive several properties regarding signal design.

A. PROPERTY 1

The performance of the optimum receiver is minimized by choosing the complex envelope of the signal, $\tilde{S}_d(t_\alpha)$, equal to the eigenfunction of the noise with the largest eigenvalue.

The proof of this statement is as follows. For any threshold Λ_0, the performance of the optimum receiver is monotone in d_o^2, where

$$d_o^2 = \int_{-T}^{T} \tilde{q}^* (t)\, S_d(t)\, dt \tag{V-7}$$

and

$$\tilde{S}_d(t_\alpha) = \int_{-T}^{T} \tilde{R}_{n_T}(t_\alpha, t_\beta)\, q(t_\beta)\, dt_\beta \qquad -T < t_\alpha < T \tag{V-8}$$

Expand $\tilde{q}\,(t_\beta)$ using the eigenfunctions[/] of $\tilde{R}_{n_T}(t_\alpha, t_\beta)$

$$\tilde{q}(t_\beta) = \sum_{i=1}^{\infty} q_i \phi_i\,(t_\beta) \tag{V-9}$$

where the ϕ_i (t) satisfy the integral equation

$$\sigma_k^2 \phi_k\,(t_\alpha) = \int_{-T}^{T} \tilde{R}_{n_T}(t_\alpha, t_\beta)\, \phi_k\,(t_\beta)\, dt_\beta \tag{V-10}$$

Substituting Equation V-9 into Equation V-8 and using Equation V-10, we have:

$$\tilde{S}_d\,(t_\alpha) = \sum_{i=1}^{\infty} q_k \sigma_k^2 \phi_k\,(t) \tag{V-11}$$

/ See Reference 7.

Similarly, we could expand $\widetilde{S}_d(t_\alpha)$ using the eigenfunctions

$$\widetilde{S}_d(t_\alpha) = \sum_{i=1}^{\infty} S_k \, \phi_k(t) \qquad \text{(V-12)}$$

Equations V-11 and V-12 imply

$$q_k = \frac{S_k}{\sigma_k^2} \qquad \text{(V-13)}$$

Now,

$$d_o^2 = \int_{-T}^{T} \left\{ \sum_{i=1}^{\infty} \frac{S_i^*}{\sigma_i^2} \phi_i^*(t) \right\} \left\{ \sum_{j=1}^{\infty} S_j \phi_j(t) \right\} dt \qquad \text{(V-14)}$$

Integrating and using the orthonormality of the eigenfunctions, we have:

$$d_o^2 = \sum_{i=1}^{\infty} \frac{|S_i|^2}{\sigma_i^2} \qquad \text{(V-15)}$$

Now, the sum of the $|S_i|^2$ is twice the energy in the signal.

$$2E_r = \sum_{i=1}^{\infty} |S_i|^2 \qquad \text{(V-16)}$$

Denote the largest eigenvalue by σ_L^2. Then clearly d_o^2 is minimized by setting:

$$|S_L|^2 = 2E_r \qquad \text{(V-17)}$$

$$S_1 = 0 \qquad i \neq L \qquad \text{(V-18)}$$

B. PROPERTY 2

If $\widetilde{S}_d(t)$ is chosen to give the minimum value of d_o^2, the optimum receiver is identical to the conventional receiver.

If, $$\widetilde{S}_d(t_\alpha) = \sqrt{2E_r}\, \phi_L(t_\alpha) \qquad \text{(V-19)}$$

Then, Equation V-8 becomes:

$$\sqrt{2E_r}\ \phi_L(t_\alpha) = \int_{-T}^{T} \widetilde{R}_{n_T}(t_\alpha, t_\beta)\ \widetilde{q}(t_\beta)\ dt_\beta \tag{V-20}$$

From Equation V-9, we see that:

$$\widetilde{q}(t_\beta) = \frac{\sqrt{2E_r}}{\sigma_L^2}\ \phi_L(t_\beta) = \frac{1}{\sigma_L^2}\ \widetilde{S}_d(t_\beta) \tag{V-21}$$

which is, of course, the conventional receiver.

C. PROPERTY 3

If the conventional receiver is used for all signals, $\widetilde{S}_d(t)$, the minimum performance is obtained when $\widetilde{S}_d(t)$ is equal to the eigenfunction of the noise with the largest eigenvalue.

The proof is analogous to that of Property 1.

D. PROPERTY 4

If $\widetilde{R}_{n_R}(t_\alpha, t_\beta)$ is a positive-definite correlation function corresponding to a process with finite variance and the $\phi_k(t)$ are a complete orthonormal set, then there is no unique $\widetilde{S}_d(t)$ that maximizes d_o^2.

The positive-definiteness implies there are no zero eigenvalues in the expansion of the noise. Since the $\phi_k(t)$ are a CON≠ set there is an infinite number of eigenvalues. Thus, there is no smallest one. This means that one can successively increase d_o^2 by choosing $\widetilde{S}_d(t)$ equal to eigenfunctions with successively smaller eigenvalues.

≠ Complete orthonormal.

The intuitive meaning of this statement should be clear. Suppose $R_{n_A}(t_1, t_1) = 0$. Then, if we consider the typical noise spectrum shown in Figure 10, we see

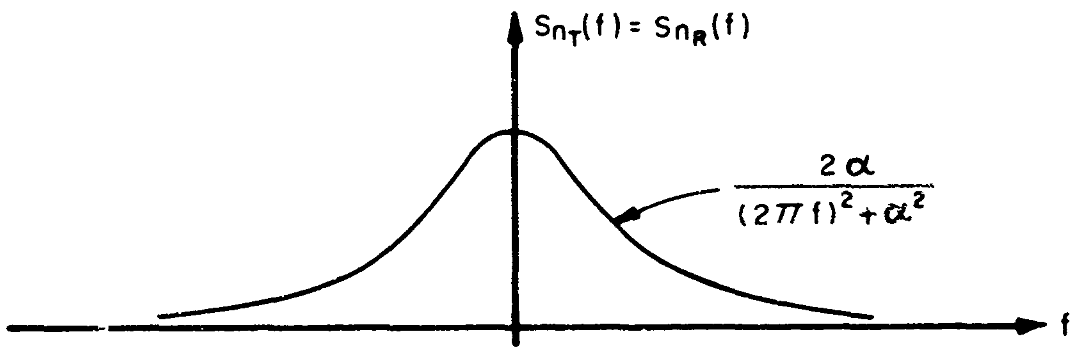

FIGURE 10 TYPICAL NOISE SPECTRUM

that the eigenfunctions are of the form:[/]

$$\phi_n(t) = k_n \cos \alpha b_n t \qquad -T < t < T \tag{V-22}$$

and the eigenvalues are of the form::

$$\sigma_n^2 = \frac{1}{4\alpha(1 + b_n^2)} \tag{V-23}$$

Where

$$b_1^2 < b_2^2 < b_3^2 < \cdots < b_n^2 \tag{V-24}$$

Thus,

$$\sigma_1^2 > \sigma_2^2 > \cdots > \sigma_n^2 \tag{V-25}$$

Now, as we take successively smaller eigenvalues, the corresponding eigenfunctions are cosines of successively increasing frequency. Thus, we can make the system perform arbitrarily well by transmitting a signal of arbitrarily high frequency.

/ See pp. 99-101 of Reference 7. Note that there are also eigenfunctions of the form $k_n \sin \alpha b_n t$. An identical argument holds for these eigenfunctions.

This fact was obvious from the shape of the noise spectrum. The limitation here is a practical one rather than a mathematical one.

Observe that if we allow processes with infinite variance, Property 4 is not true.

As an example, consider the spectrum shown in Figure 11

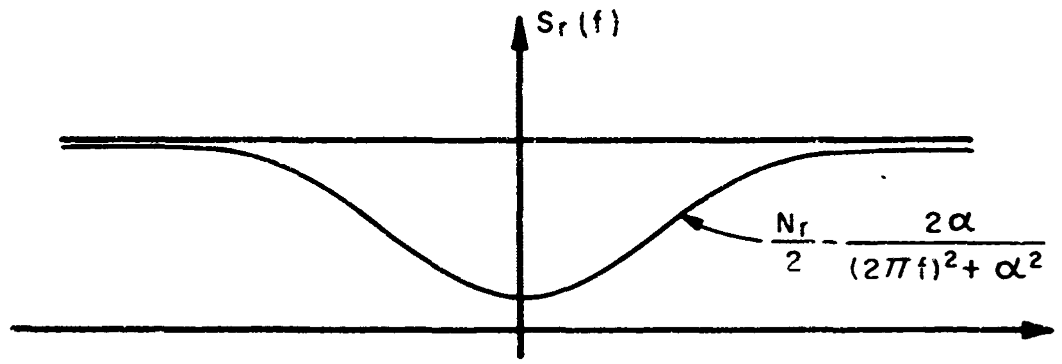

FIGURE 11 AN EXAMPLE OF NOISE SPECTRUM WHEN PROCESSES HAVE INFINITE VARIANCE

This spectrum has a smallest eigenvalue,

$$\sigma^2_{min} = \frac{N_r}{2} - \sigma^2_1 = \frac{N_r}{2} - \frac{1}{4\alpha\,(1 + b^2_1)} \tag{V-26}$$

However, since this type of spectrum will not arise in our work, Property 4 will be valid.

In our problem, the noise correlation function is determined by the transmitted signal.

$$\tilde{R}_{n_t}(t_1, t_2) = \tilde{R}_{n_A}(t_1, t_2) + \tilde{R}_{n_R}(t_1, t_2) \tag{V-27}$$

$$\tilde{R}_{n_T}(t_1, t_2) = 2N_o u_o(t_1 - t_2) + \int_{-T}^{T} a(x)\, M_{\omega_q|x}(t_\alpha - t_\beta : x)\,\tilde{S}(t_\alpha - x)\,\tilde{S}^*(t_\beta - x)\, dx \tag{V-28}$$

Thus, as one changes the signal shape, one changes $\tilde{R}_{n_T}(t_1, t_2)$. Therefore the problem of finding the worst signal shape is difficult for the general case. Moreover, when the correlation function is specified by Equation V-28, one can not always satisfy Equation V-19. We will confine ourselves temporarily to some simple scattering functions and signal shapes. Clearly, the reason we are concerned with the worst signal shape is that it tells us what characteristics we want to avoid in our signal. After finding the minimum, we can show how the performance improves as we move away from the minimum.

In Section VI, we will consider a stationary reverberation return. In Section VII, we will consider a non-stationary return. In each case, we will evaluate the performance of the optimum and the conventional receiver as a function of the signal shape and scattering function.

VI. GAUSSIAN SIGNALS, UNIFORM REVERBERATION

A. ASSUMPTIONS OF THE MODEL

In this section, we apply the results of the preceding sections to a specific situation. From our discussion in Sections III and IV, we observe that to evaluate the performance of the conventional receiver in a uniform reverberation environment we need the signal shape and the distribution in Doppler of the scatterers.

1. Signal and Receiver Properties

We will assume that the velocity of each scatter r is a zero-mean Gaussian random variable.

Thus,

$$p_{\omega_D}(X) = \frac{1}{\sqrt{2\pi}\,B}\, e^{-\frac{X^2}{2B^2}} \qquad \text{(VI-1)}$$

where B is the rms Doppler shift.†

To use Equation III-36, we require the characteristic function:

$$M_{\omega_D}(j\tau) = e^{-\frac{B^2\tau^2}{2}} \qquad \text{(VI-2)}$$

We will assume that the transmitted signal has a Gaussian envelope and linear frequency modulation.

Thus,

$$S_T(t) = \mathrm{Re}\left[k\ e^{-at^2-jbt^2}\ e^{j\omega_c t}\right] \qquad \text{(VI-3)}$$

The complex envelope is:

$$\tilde{f}(t) = k\,e^{-at^2-jbt^2} \qquad \text{(VI-4)}$$

---

† Observe that B depends on the rms scatterer velocity and the carrier frequency of the transmitted signal.

The amplitude and instantaneous frequency are shown in Figure 12. We observe that the pulse duration is infinite. This is an idealization which makes the analysis appreciably simpler.

Since

$$2E_t = \int_{-\infty}^{\infty} |f(t)|^2 dt = k^2 \int_{-\infty}^{\infty} e^{-2at^2} dt = k^2 \frac{\sqrt{2\pi}}{2\sqrt{a}} \qquad \text{(VI-5)}$$

We have:

$$k^2 = 4E_t \left(\frac{a}{2\pi}\right)^{\frac{1}{2}} = 2E_t \left(\frac{2a}{\pi}\right)^{\frac{1}{2}} \qquad \text{(VI-6)}$$

To evaluate the performance, we need the Fourier transform of the complex envelope.

$$F(j\omega) = 2E_t^{\frac{1}{2}} \left(\frac{a}{2\pi}\right)^{\frac{1}{4}} \int_{-\infty}^{\infty} e^{-at^2 - jbt^2 + j\omega t} dt \qquad \text{(VI-7)}$$

Completing the square:

$$F(j\omega) = \exp - \frac{\omega^2}{4(a+jb)^2} \; 2E_t^{\frac{1}{2}} \left(\frac{a}{2\pi}\right)^{\frac{1}{4}} \int_{-\infty}^{\infty} \exp - (a+jb) \left[t^2 - \frac{j\omega}{a+jb} t - \frac{\omega^2}{4(a+jb)^2} \right] dt$$

(VI-8)

Integrating, we have:

$$F(j\omega) = 2E_t^{\frac{1}{2}} \left(\frac{a}{2\pi}\right)^{\frac{1}{4}} \cdot \sqrt{2\pi} \; \frac{1}{(2(a+jb))^{\frac{1}{2}}} \exp - \frac{\omega^2}{4(a+jb)^2}$$

$$= 2E_t^{\frac{1}{2}} (2\pi)^{\frac{1}{4}} \left[\frac{a^{\frac{1}{2}}}{2(a^2+b^2)^{\frac{1}{2}}} \right]^{\frac{1}{2}} \exp - \frac{\omega^2 a}{4(a^2+b^2)} + \frac{\omega^2 \cdot jb}{4(a^2+b^2)} - \frac{1}{2} j \tan^{-1} \frac{b}{a}$$

(VI-9)

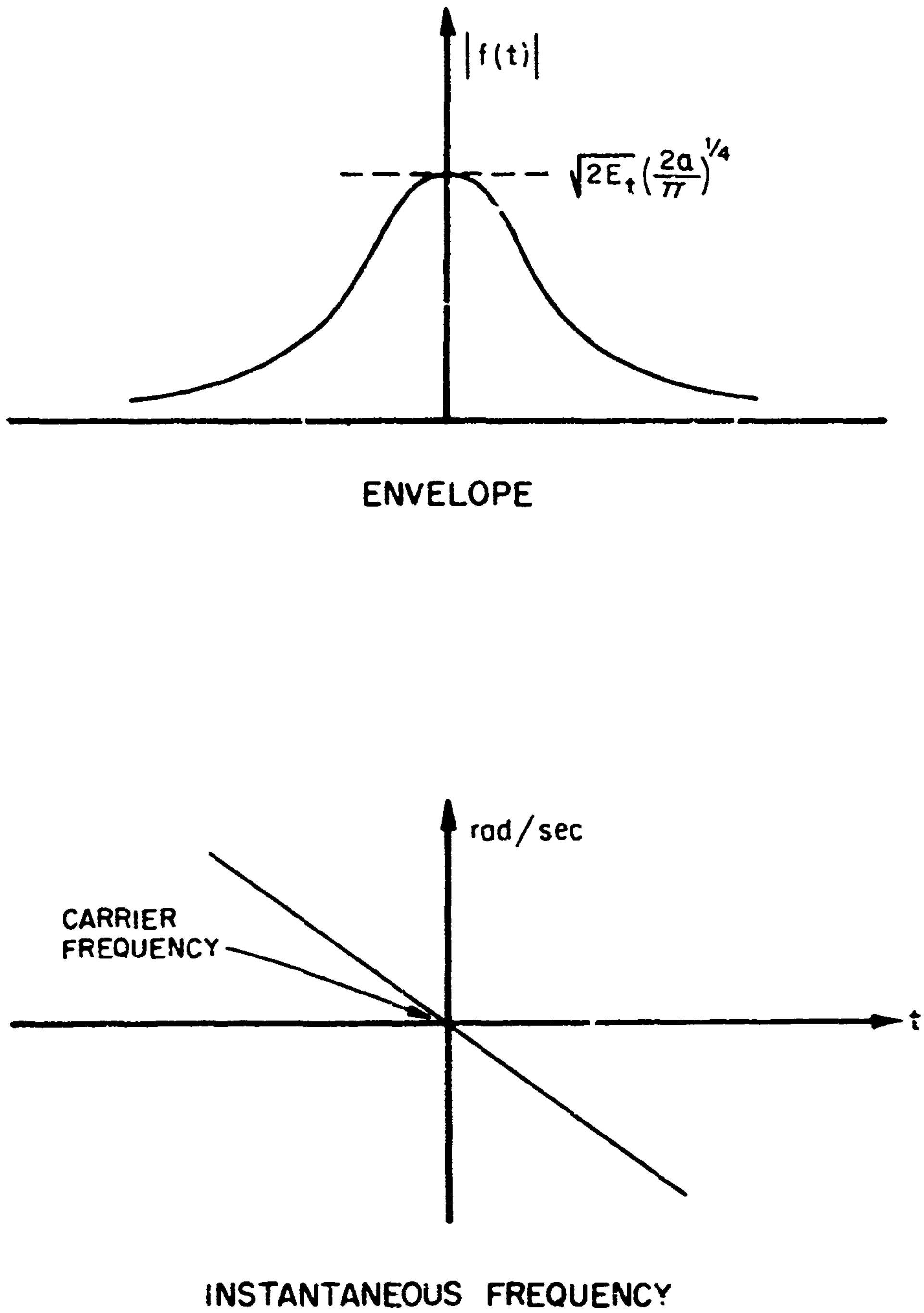

FIGURE 12 MAGNITUDE AND PHASE OF COMPLEX ENVELOPE

If we define:

$$\Delta = \frac{(a^2 + b^2)}{a} \qquad \text{(VI-10)}$$

we may write:

$$F(j\omega) = 2E_t^{\frac{1}{2}} (2\pi)^{\frac{1}{4}} \frac{1}{(2\Delta^{\frac{1}{2}})^{\frac{1}{2}}} \exp - \frac{\omega^2}{4\Delta} + j\left[\omega^2 \frac{b}{4(a^2+b^2)} - \tfrac{1}{2}\tan^{-1}\frac{b}{a}\right]$$

(VI-11)

The magnitude squared is:

$$\left|F(j\omega)\right|^2 = 4E_t \cdot (2\pi)^{\frac{1}{2}} \cdot \frac{1}{2\Delta^{\frac{1}{2}}} \exp - \frac{\omega^2}{2\Delta} = 2E_t \cdot \left(\frac{2\pi}{\Delta}\right)^{\frac{1}{2}} \exp - \frac{\omega^2}{2\Delta}$$

(VI-12)

As a simple check on the constant, we observe that:

$$\int_{-\infty}^{\infty} \left|F(j\omega)\right|^2 \frac{d\omega}{2\pi} = 2E_t \qquad \text{(IV-13)}$$

Now, the correlation function $R_f(\tau)$ is simply the inverse transform:

$$\widetilde{R}_f(\tau) = \int_{-\infty}^{\infty} 2E_t \left(\frac{2\pi}{\Delta}\right)^{\frac{1}{2}} \exp - \frac{\omega^2}{2\Delta} \exp - j\omega\tau \frac{d\omega}{2\pi} = 2E_t \cdot \exp - \frac{\Delta\tau^2}{2}$$

(VI-14)

From Equation IV-65, we have (for the stationary case):

$$R_{n_R}(\tau) = \frac{I_{av}}{2} M_{\omega_D}(\tau)\, R_f(\tau) \qquad \text{(VI-15)}$$

Substituting Equations VI-1 and VI-14 into Equation VI-15, we have:

$$R_{n_R}(\tau) = E_t \cdot I_{av} \exp - \frac{1}{2}(B^2 + \Delta)\tau^2 \qquad \text{(VI-16)}$$

Let $$\gamma^2 = B^2 + \Delta \qquad \text{(VI-17)}$$

Then,

$$R_{n_r}(\tau) = E_t \cdot I_{av} \exp - \frac{\gamma^2 \tau^2}{2} \tag{VI-18}$$

and the reverberation spectrum is:

$$S_{n_R}(\omega) = \frac{2E_t \cdot I_{av}}{2} \frac{\sqrt{2\pi}}{\gamma} \exp - \frac{\omega^2}{2\gamma^2} \tag{VI-19}$$

Using the conventional matched filter, we have:

$$\left|F(j\omega - j\omega_D)\right|^2 = 2E_r \frac{\sqrt{2\pi}}{\Delta^{\frac{1}{2}}} \exp - \frac{(\omega - \omega_D)^2}{2\Delta} \tag{VI-20}$$

2. Evaluation of Conventional Receiver

To evaluate the conventional receiver, we simply substitute Equations VI-19 and VI-20 into Equation IV-72. This gives:

$$d_c^2 = \frac{\left(\frac{2E_r}{N_o}\right)^2}{\frac{2E_r}{N_o} + \frac{1}{N_o^2} \int_{-\infty}^{\infty} 2E_r \cdot \frac{\sqrt{2\pi}}{\Delta^{\frac{1}{2}}} \exp - \frac{(\omega - \omega_D)^2}{2} \cdot \frac{2E_t \cdot I_{av}}{2} \frac{\sqrt{2\pi}}{\gamma} \exp - \frac{\omega^2}{2\gamma^2} \frac{d\omega}{2\pi}} \tag{VI-21}$$

or

$$d_c^2 = \frac{2E_r}{N_o} \left\{ 1 + \frac{E_t I_{av}}{N_o} \cdot \frac{1}{\Delta^{\frac{1}{2}} \gamma} \int_{-\infty}^{+\infty} \exp - \left\{ \frac{(\omega - \omega_D)^2}{2\Delta} + \frac{\omega^2}{2\gamma^2} \right\} d\omega \right\}^{-1} \tag{VI-22}$$

Collecting terms and completing the square, we have:

$$d_c^2 = \frac{\frac{2E_r}{N_o}}{\frac{\sqrt{2\pi}}{\Delta^{\frac{1}{2}}} \int_{-\infty}^{\infty} \left| F(j\omega - j\omega_D) \right|^2 \left[1 + 2E_t \cdot I_{av} \frac{\sqrt{2\pi}}{\gamma} \exp - \frac{\omega^2}{2\gamma^2} \right] \frac{d\omega}{2\pi}} \qquad \text{(VI-23)}$$

where

$$\gamma^2 = B^2 + \Delta \qquad \text{(VI-24)}$$

Evaluating the denominator, we have:

$$\text{den.} = 1 + \frac{E_t \cdot I_{av}}{N_o \Delta^{\frac{1}{2}} \gamma} \int_{-\infty}^{\infty} \exp - \frac{(\omega - \omega_D)^2}{2\Delta} \exp - \frac{\omega^2}{2\gamma^2} d\omega \qquad \text{(VI-25)}$$

Letting $\mathcal{E} = \frac{I}{N_{o/2}}$ (VI-26)

and collecting terms and completing the square, we have:

$$\text{den.} = 1 + \frac{E_t \mathcal{E}}{2N_o \Delta^{\frac{1}{2}} \gamma} \int_{-\infty}^{\infty} \exp - \frac{B^2 + 2\Delta}{2\Delta(B^2 + \Delta)} \left[\omega - \omega_D \frac{B^2 + \Delta}{B^2 + 2\Delta} \right]^2 d\omega$$

$$\cdot \exp - \frac{\omega_D^2}{2(B^2 + 2\Delta)} \qquad \text{(VI-27)}$$

Integrating, we have:

$$\text{den.} = 1 + \frac{E_t \mathcal{E}}{2\Delta^{\frac{1}{2}} \cdot \gamma} \frac{\sqrt{2\pi}\, \Delta^{\frac{1}{2}} (B^2 + \Delta)^{\frac{1}{2}}}{(B^2 + 2\Delta)^{\frac{1}{2}}} \exp - \frac{\omega_D^2}{2(B^2 + 2\Delta)} \qquad \text{(VI-28)}$$

This expression reduces to:

$$\text{den.} = 1 + \frac{E_t B \sqrt{2\pi}}{2(B^2 + 2\Delta)^{\frac{1}{2}}} \exp - \frac{\omega_D^2}{2(B^2 + 2\Delta)} \qquad \text{(VI-29)}$$

Now consider the behavior as a function of ω_D, B, and Δ.

For $B \neq 0$, this can be rewritten as:

$$\text{den.} = 1 + \frac{E_t B \sqrt{2\pi}}{2B} \cdot \frac{1}{\left(1 + \frac{2\Delta}{B^2}\right)^{\frac{1}{2}}} \exp - \frac{\omega_D^2}{2B^2\left(1 + \frac{2\Delta}{B^2}\right)} \qquad \text{(VI-30)}$$

We observe that the ratios of importance are:

$\frac{\omega_D}{B}$: The ratio of the target Doppler shift to the rms reverberation Doppler shift

and

$\frac{\sqrt{2}\Delta^{\frac{1}{2}}}{B}$: The ratio of the effective bandwidth of the signal to the rms reverberation Doppler shift.

To maximize d_c^2, we want to minimize the second term in Equation VI-30. The first coefficient is a function of the environment and the transmitted energy. Considering this constant, we want to study the behavior of the function:

$$f\left(\frac{\omega_D}{B}, \frac{\sqrt{2}\,\Delta^{\frac{1}{2}}}{B}\right) = \left(1 + \frac{2\Delta}{B^2}\right)^{\frac{1}{2}} \exp + \frac{\omega_D^2}{2B^2\left(1 + \frac{2\Delta}{B^2}\right)} \qquad \text{(VI-31)}$$

The curve is shown in Figure 13.[†]

[†] One observes the similarity between Figure 13 and Figure 1 of Reference 14.

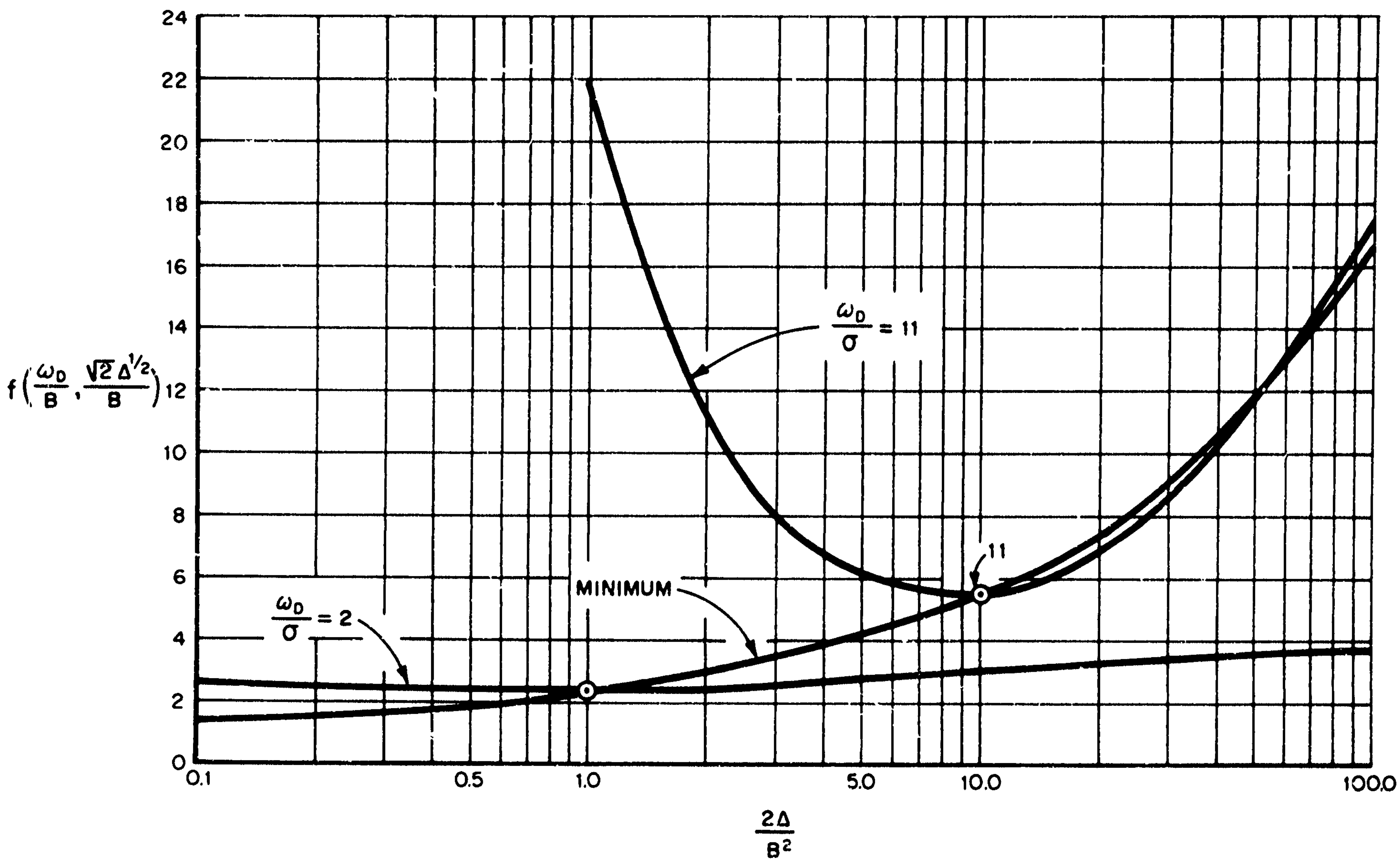

FIGURE 13 PLOT OF $f\left(\frac{\omega_D}{B}, \frac{\sqrt{2}\,\Delta^{\frac{1}{2}}}{B}\right)$

As pointed out in Section V, there will not be a unique maximum.

The minimum value of $f\left(\frac{\omega_D}{B}, \frac{\sqrt{2}\,\Delta^{\frac{1}{2}}}{B}\right)$ occurs when

$$\frac{2\Delta}{B^2} = \frac{\omega_D^2}{B^2} - 1 \qquad \omega_D < B \qquad \text{(VI-32)}$$

$$= 0 \qquad \omega_D < B$$

At the minimum:

$$f\left(\frac{\omega_D}{B}, \frac{\sqrt{2}\,\Delta^{\frac{1}{2}}}{B}\right) = 1.65\,\frac{\omega_T}{B} = 1.65\left(1 + \frac{2\Delta}{B^2}\right)^{\frac{1}{2}} \qquad \text{(VI-33)}$$

Clearly the importance of the term given by Equation VI-31 depends on the value of its coefficient in Equation VI-30.

To study this effect, we substitute Equation VI-30 into Equation VI-22:

$$d_{nc}^2 = \frac{d_c^2}{2E_{r/N_o}} = \cfrac{1}{1+D\,\cfrac{1}{\left(1+\cfrac{2\Delta}{B^2}\right)^{\frac{1}{2}}}\exp - \cfrac{\omega_D^2}{2B^2\left(1+\cfrac{2\Delta}{B^2}\right)}} \qquad \text{(VI-34)}$$

where

$$D = \frac{E_t\,\beta\sqrt{2\pi}}{2B} \qquad \text{(VI-35)}$$

Physically this represents the reverberation-to-ambient-noise level in the reverberation bandwidth.

Recall that $\beta = \frac{I}{N_{o/2}}$. Thus, $D = \frac{E_t I}{2} \Big/ \frac{N_o}{2}\,\frac{\beta}{\sqrt{2\pi}}$

The denominator is simply the noise power out of a filter with a Gaussian spectrum when the input is white noise of spectral height $N_{o/2}$. From Equation III-32, we observe that the numerator is just the total received power. Thus, D is the reverberation-to-ambient-noise ratio.

The physical meaning of d^2_{nc} should be clear. In the absence of reverberation it equals 1.0. A decrease to a value of less than 1.0 represents the loss in detectability due to reverberation.

We have plotted d^2_{nc} for the following parameters:

D	Physical Meaning	Figure
0.3	reverberation < ambient noise	14
1.0	reverberation = ambient noise	15
10.0	rev./ambient noise = 10 db	16
100.0	rev./ambient noise = 20 db	17

The parameters on the curves are $w_{D/B}$. This is the ratio of the target velocity to the reverberation Doppler.

The horizontal axis is $\Delta^{\frac{1}{2}}/B$. This is the ratio of the signal bandwidth to the reverberation Doppler.

Several observations may be made with respect to this class of signals:

a. For zero target velocities, we have monotone improvement as the bandwidth increases.

b. For non-zero target velocities, one can use either very small or very large bandwidth signals. The point of the exact minimum is a function of $w_{D/B}$ as given by Equation VI-33.

c. For small Δ, the non-zero target velocities, the improvement increases rapidly. However, for large Δ, <u>all</u> targets behave the same./

Now we consider the optimum receiver.

/ This statement is, of course, for the class of signals that we are considering (i.e., linear FM and Gaussian envelopes). It should be emphasized that this statement is <u>not</u> true for <u>all</u> signals.

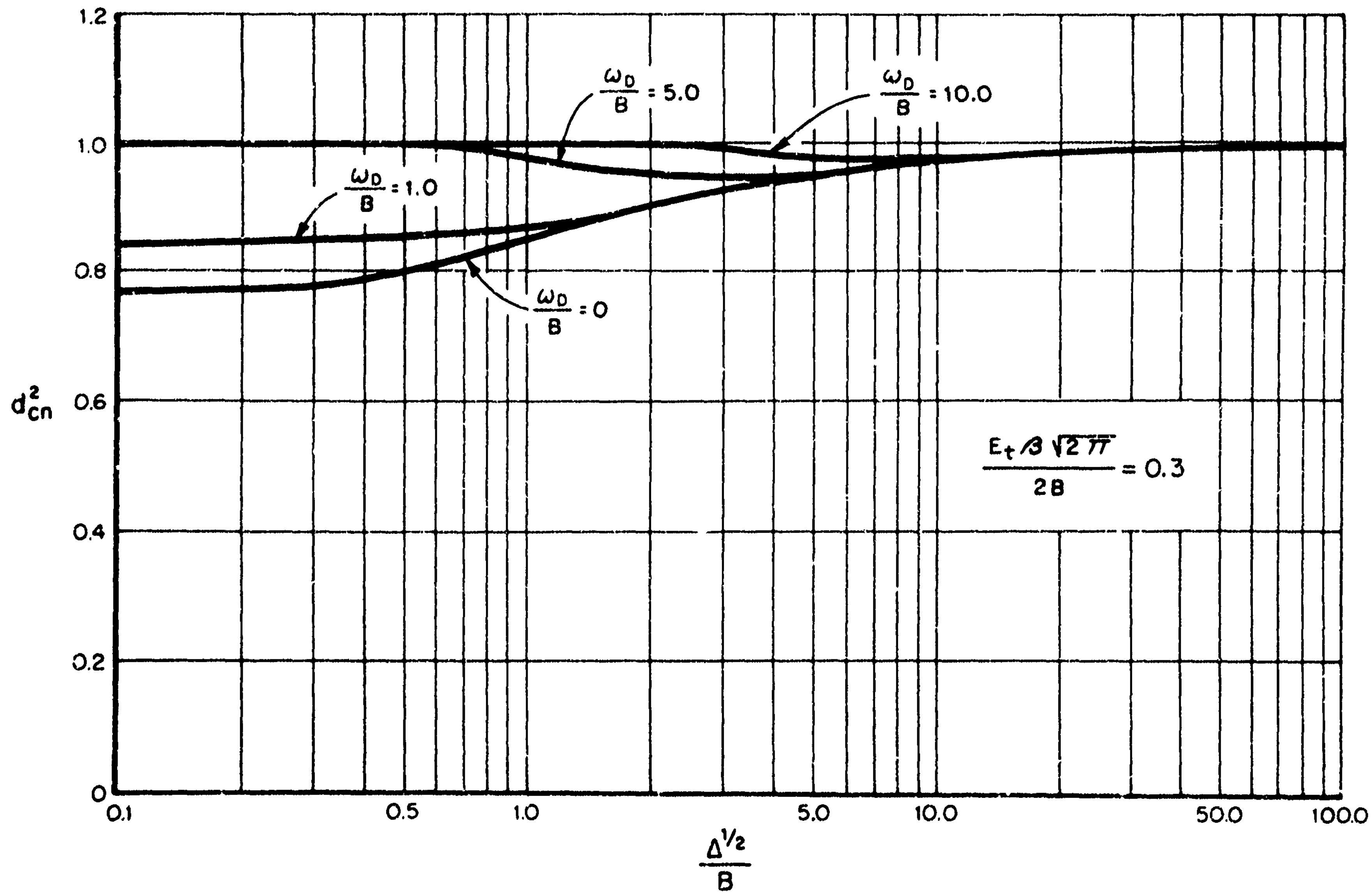

FIGURE 14 LOSS IN DETECTABILITY DUE TO REVERBERATION (D = 0.3)

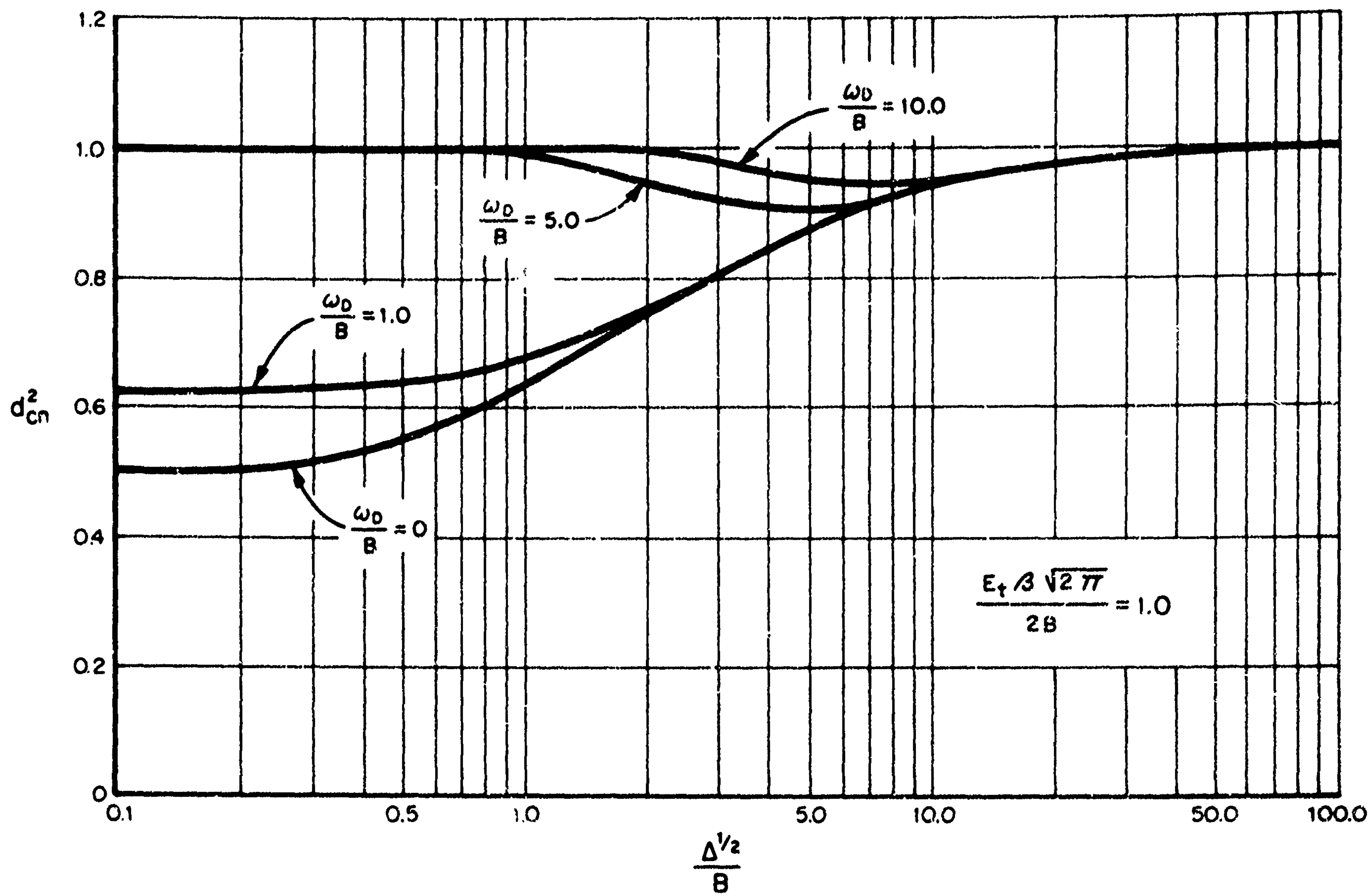

FIGURE 15 LOSS IN DETECTABILITY DUE TO REVERBERATION (D = 1.0)

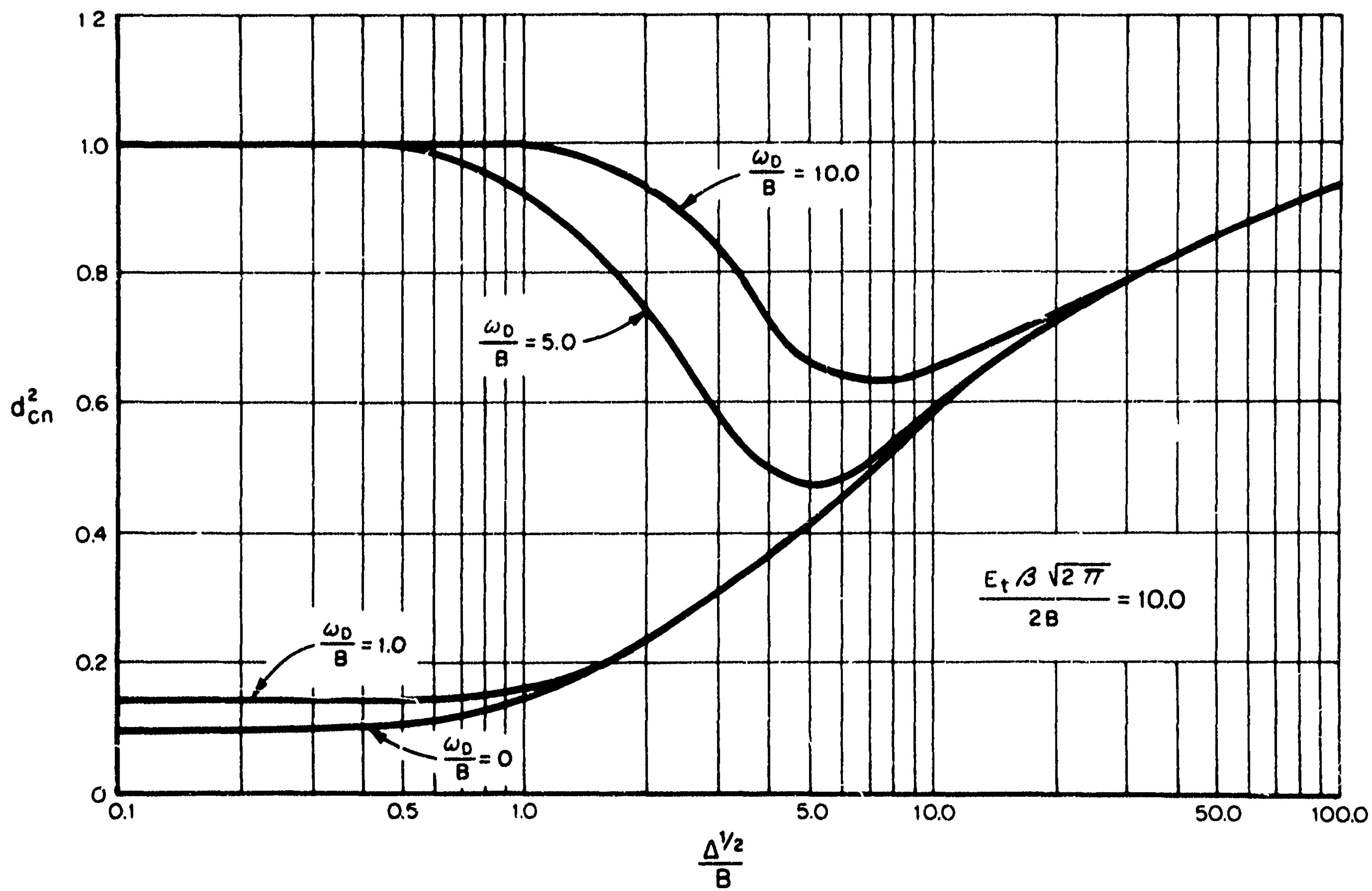

FIGURE 16 LOSS IN DETECTABILITY DUE TO REVERBERATION (D = 10.0)

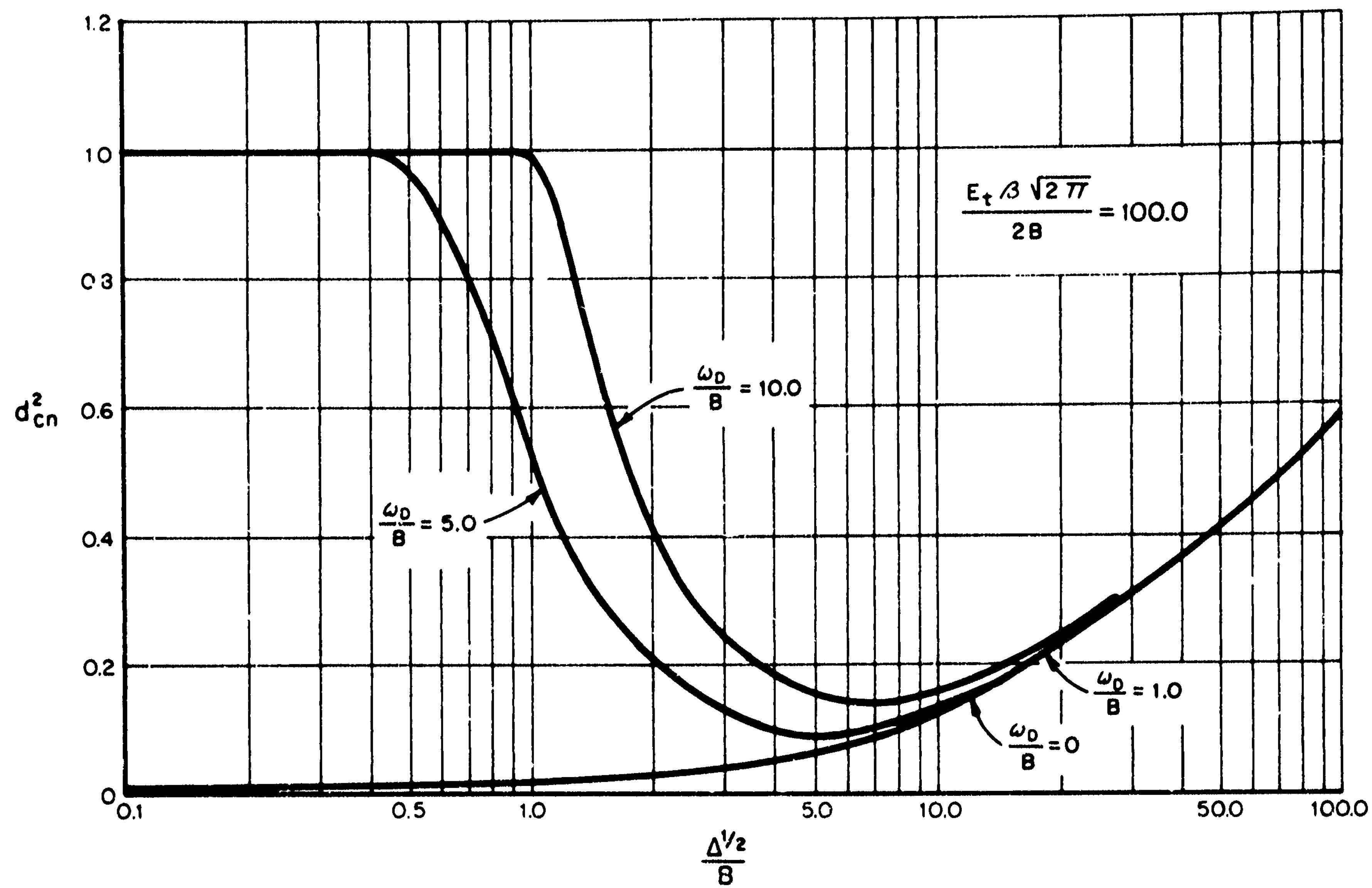

FIGURE 17 LOSS IN DETECTABILITY DUE TO REVERBERATION (D = 100.0)

B. PERFORMANCE OF OPTIMUM RECEIVER

We will use the same assumptions as in the preceding section. Now, from Equation IV-46, we have

$$d_o^2 = \int_{-\infty}^{\infty} \frac{|S_d(\omega)|^2}{N_o + S_{n_r}^*(\omega)} \frac{d\omega}{2\pi} \tag{VI-36}$$

Substituting Equations VI-19 and VI-20 into VI-36, we have:

$$d_o^2 = \int_{-\infty}^{\infty} \frac{2E_r \frac{\sqrt{2\pi}}{\Delta^{\frac{1}{2}}} \exp\left\{-\frac{(\omega-\omega_D)^2}{2\Delta}\right\}}{N_o + E_t I \frac{\sqrt{2\pi}}{\gamma} \exp\left\{-\frac{\omega^2}{2\gamma^2}\right\}} \frac{d\omega}{2\pi} \tag{VI-37}$$

This can be rewritten as:

$$d_o^2 = \frac{2E_r}{N_o} \int_{-\infty}^{\infty} \frac{\frac{\sqrt{2\pi}}{\Delta^{\frac{1}{2}}} \exp\left\{-\frac{(\omega-\omega_D)^2}{2\Delta}\right\}}{1 + \frac{E_t \beta}{2} \frac{\sqrt{2\pi}}{\gamma} \exp\left\{-\frac{\omega^2}{2\gamma^2}\right\}} \frac{d\omega}{2\pi} \tag{VI-38}$$

For arbitrary parameter values, one cannot obtain a closed form solution. We will first consider the case where the reverberation is small.

1. Perturbation Solution (Low Reverberation Levels)

Consider the case when

$$\frac{E_t \beta}{2} \frac{\sqrt{2\pi}}{\gamma} << 1 \tag{VI-39}$$

In this, the denominator can be expanded in a series which is absolutely convergent.

Then,

$$d_o^2 = \frac{2E_t}{N_o} \quad \frac{1}{\sqrt{2\pi}\,\Delta^{\frac{1}{2}}} \int_{-\infty}^{\infty} \exp\left\{-\frac{(\omega-\omega_D)^2}{2\Delta}\right\}$$

$$\cdot\left\{1-\left(\frac{E_t\cdot B}{2}\,\frac{\sqrt{2\pi}}{\gamma}\right)\exp\left\{-\frac{\omega^2}{2\gamma^2}\right\}+\left(\frac{E_t\cdot B}{2}\,\frac{\sqrt{2\pi}}{\gamma}\right)^2\exp\left\{-\frac{2\omega^2}{2\gamma^2}\right\}\right.$$

$$\left.+\ldots(-1)^n\left(\frac{E_t\cdot B}{2}\,\frac{\sqrt{2\pi}}{\gamma}\right)^n\exp\left\{-\frac{n\omega^2}{2\gamma^2}\right\}+\ldots\right\}d\omega \qquad \text{(VI-40)}$$

The first term is just $\frac{2E_r}{N_o}$, the conventional filter result.

The second term is

$$d_o^2\,(2) = \frac{2E_r}{N_o}\cdot\frac{1}{\sqrt{2\pi}\,\Delta^{\frac{1}{2}}}\cdot\frac{E_t\cdot B}{2}\,\frac{\sqrt{2\pi}}{\gamma}\int_{-\infty}^{\infty}\exp\left\{-\frac{(\omega-\omega_D)^2}{2\Delta}-\frac{\omega^2}{2\gamma^2}\right\}d\omega \qquad \text{(VI-41)}$$

Completing the square and integrating, we have:

$$d_o^2\,(2) = \frac{-2E_r}{N_o}\cdot\frac{1}{\sqrt{2\pi}\,\Delta^{\frac{1}{2}}}\cdot\frac{E_t\cdot B}{2}\,\frac{\sqrt{2\pi}}{\gamma}\cdot\frac{\Delta^{\frac{1}{2}}\gamma}{(B^2+2\Delta)^{\frac{1}{2}}}\sqrt{2\pi}\exp-\frac{\omega_D^2}{2(B^2+2\Delta)}$$

$$=\left(\frac{2E_r}{N_o}\right)\cdot\frac{E_t\cdot B}{2}\,\frac{\sqrt{2\pi}}{(B^2+2\Delta)^{\frac{1}{2}}}\exp-\frac{\omega_D^2}{2(B^2+2\Delta)} \qquad \text{(VI-42)}$$

Similarly, the (n+1)st term is:

$$d_o^2(n+1) = \frac{2E_r}{N_o}\cdot\frac{1}{\sqrt{2\pi}\Delta^{\frac{1}{2}}}\left(\frac{E_t\cdot B}{2}\cdot\frac{\sqrt{2\pi}}{\gamma}\right)^n\int_{-\infty}^{\infty}\exp\left\{-\frac{(\omega-\omega_D)^2}{2\Delta}-\frac{n\omega^2}{2\gamma^2}\right\}d\omega \qquad \text{(VI-43)}$$

Completing the square and integrating, we have:

$$d_o^2\,(n+1) = \frac{2E_r}{N_o}\left(\frac{E_t\beta}{2}\;\frac{\sqrt{2\pi}}{\gamma}\right)^n \frac{(-1)^n\,\gamma}{\left(B^2+(n+1)\Delta\right)^{\frac{1}{2}}}\exp -\frac{\omega_D^2}{2}\,\frac{n}{B^2+(n+1)\Delta} \tag{VI-44}$$

Then,

$$d_o^2 = \frac{2E_r}{N_o}\sum_{n=0}^{\infty}(-1)^n\left(\frac{E_t\beta}{2}\;\frac{\sqrt{2\pi}}{\gamma}\right)^n \frac{\gamma}{\left(B^2+(n+1)\Delta\right)^{\frac{1}{2}}}\exp -\frac{\omega_D^2}{2}\,\frac{n}{B^2+(n+1)\Delta} \tag{VI-45}$$

Using the same type normalization as previously we can write: (for $B\neq 0$)

$$d_o^2 = \frac{2E_r}{N_o}\left\{1-\left(\frac{E_t\beta\sqrt{2\pi}}{2B}\right)\frac{1}{\left(1+\frac{2\Delta}{B^2}\right)^{\frac{1}{2}}}\exp -\frac{1}{2}\left(\frac{\omega_D}{B}\right)^2\frac{1}{\left(1+\frac{2\Delta}{B^2}\right)}\right.$$

$$+\cdots+(-1)^n\left(\frac{E_t\beta\sqrt{2\pi}}{2B}\right)^n\frac{1}{\left(1+\frac{\Delta}{B^2}\right)^{\frac{n-1}{2}}\left(1+\frac{(n+1)\Delta}{B^2}\right)^{\frac{1}{2}}}$$

$$\exp -\frac{1}{2}\left(\frac{\omega_D^2}{B}\right)^2\frac{n}{1+\frac{(n+1)\Delta}{B^2}} \tag{VI-46}$$

We observe that the first two terms of the series are identical to a series expansion of d_c^2

$$d_c^2 \cong \frac{2E_r}{N_o}\left\{1-\frac{E_t\beta}{2}\;\frac{\sqrt{2\pi}}{\gamma}\exp\left\{-\frac{\omega^2}{2\gamma^2}\right\}\right\}+\cdots \tag{VI-47}$$

which is the signal-to-noise ratio at the output of the conventional filter.

Several observations may be made by examining Equation VI-46:

a. As $\frac{\Delta}{B^2} \to \infty$, the performance of the conventional and that of the optimum filter will approach each other. Physically, this means that a signal whose bandwidth is much greater than the reverberation Doppler tends to eliminate the reverberation disturbance.

b. As $\frac{\omega_D}{B} \to \infty$, the performance of the conventional and that of the optimum filter will approach each other. Physically, this means that if the target Doppler is much greater than the reverberation Doppler, the effect of the reverberation is reduced.

c. As $\frac{E_t \beta}{2} \frac{\sqrt{2\pi}}{\gamma} \to 0$, the performance of the conventional and that of the optimum filter will approach each other. This is just the obvious fact that as the reverberation strength goes to zero, the optimum filter becomes the conventional filter.

These observations really indicate when the effects of reverberation are not important. Now consider the case for arbitrary reverberation levels.

2. Arbitrary Reverberation Levels

Now we consider the general case. The normalized d_{opt}^2 is:

$$d_{no}^2 \triangleq \frac{d_o^2}{\frac{2E_r}{N_o}} = \int_{-\infty}^{\infty} \frac{\frac{\sqrt{2\pi}}{\Delta^{1/2}} \exp\left\{-\frac{(\omega-\omega_D)^2}{2\Delta}\right\}}{1+\frac{E_t \beta}{2}\frac{\sqrt{2\pi}}{\gamma}\exp\left\{-\frac{\omega^2}{2\gamma^2}\right\}} \frac{d\omega}{2\pi} \qquad \text{(VI-48)}$$

Letting $x \triangleq \frac{\omega}{\Delta^{1/2}}$ and rewriting, we have:

$$d_{no}^2 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\exp\left\{-\frac{1}{2}\left[x - \frac{\omega_D/B}{\Delta^{1/2}/B}\right]^2\right\}}{1+\frac{E_t \beta \sqrt{2\pi}}{2B}\frac{1}{\left(1+\frac{\Delta}{B^2}\right)^{1/2}}\exp\left\{-\frac{1}{2}\frac{x^2}{1+\frac{B^2}{\Delta}}\right\}} dx \qquad \text{(VI-49)}$$

We observe that the quantities of importance are:

a. ω_D/B : ratio of target Doppler to reverberation Doppler

b. $\Delta^{\frac{1}{2}}/B$: ratio of effective signal bandwidth to reverberation Doppler

c. $\frac{E_t \beta \sqrt{2\pi}}{2B} \triangleq D$: ratio of the total reverberation power level to the noise power level in the reverberation bandwidth.†

The integral in Equation VI-49 was evaluated numerically for several cases. The results are shown in Figures 18 and 19. We observe that the curves have the same general characteristics as the curves for the conventional receiver. For D = 0.3 and 1.0, the quantitative difference in the performance is insignificant and the curves are the same as those in Figures 14 and 15. However, for D = 10 and 100 (i.e., reverberation/ambient noise ratios of +10 and +20 db), there are appreciable differences in some cases.

Let us examine the effects of these differences. To demonstrate the method of comparison, consider a specific example, in which the parameters are:

a. $\omega_D/B = 5.0$

b. $\Delta^{\frac{1}{2}}/B = 1.0$

c. $D = 100.0$

For this set of parameter values;

$$d_{no}^2 = 0.762 \qquad \text{(VI-50)}$$

and

$$d_{nc}^2 = 0.528 \qquad \text{(VI-51)}$$

---

† Recall that $\beta = \frac{I}{N_{o/2}}$. Thus, $D = \frac{E_t I}{2} \bigg/ \frac{N_o}{2} \frac{\beta}{\sqrt{2\pi}}$

The denominator is simply the noise power out of a filter with a Gaussian spectrum when the input is white noise of spectral height $N_{o/2}$. From Equation III-32, we observe that the numerator is just the total received power. Thus, D is the reverberation-to-ambient-noise ratio.

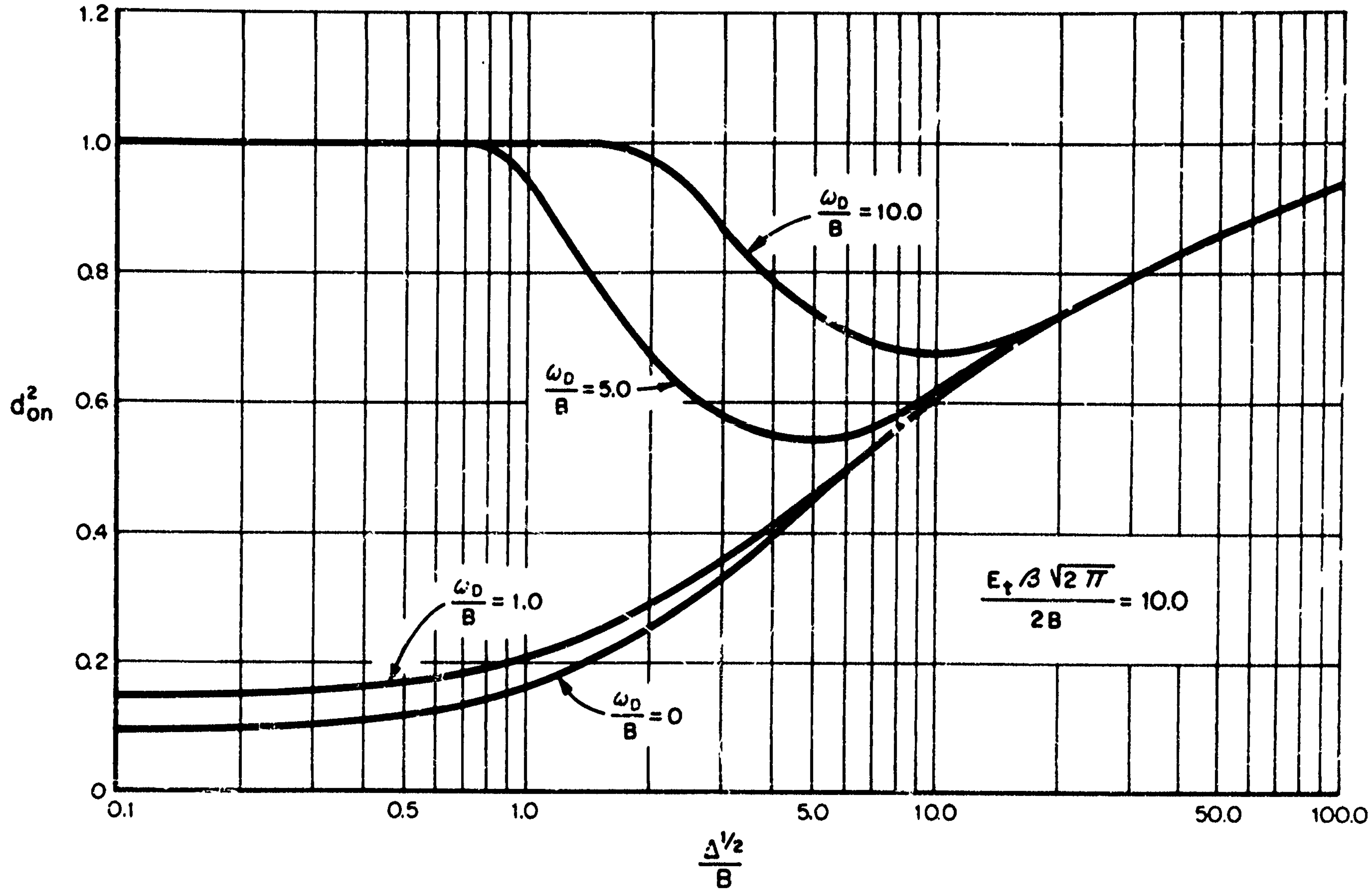

FIGURE 18 OPTIMUM RECEIVER PERFORMANCE (D= 10.0)

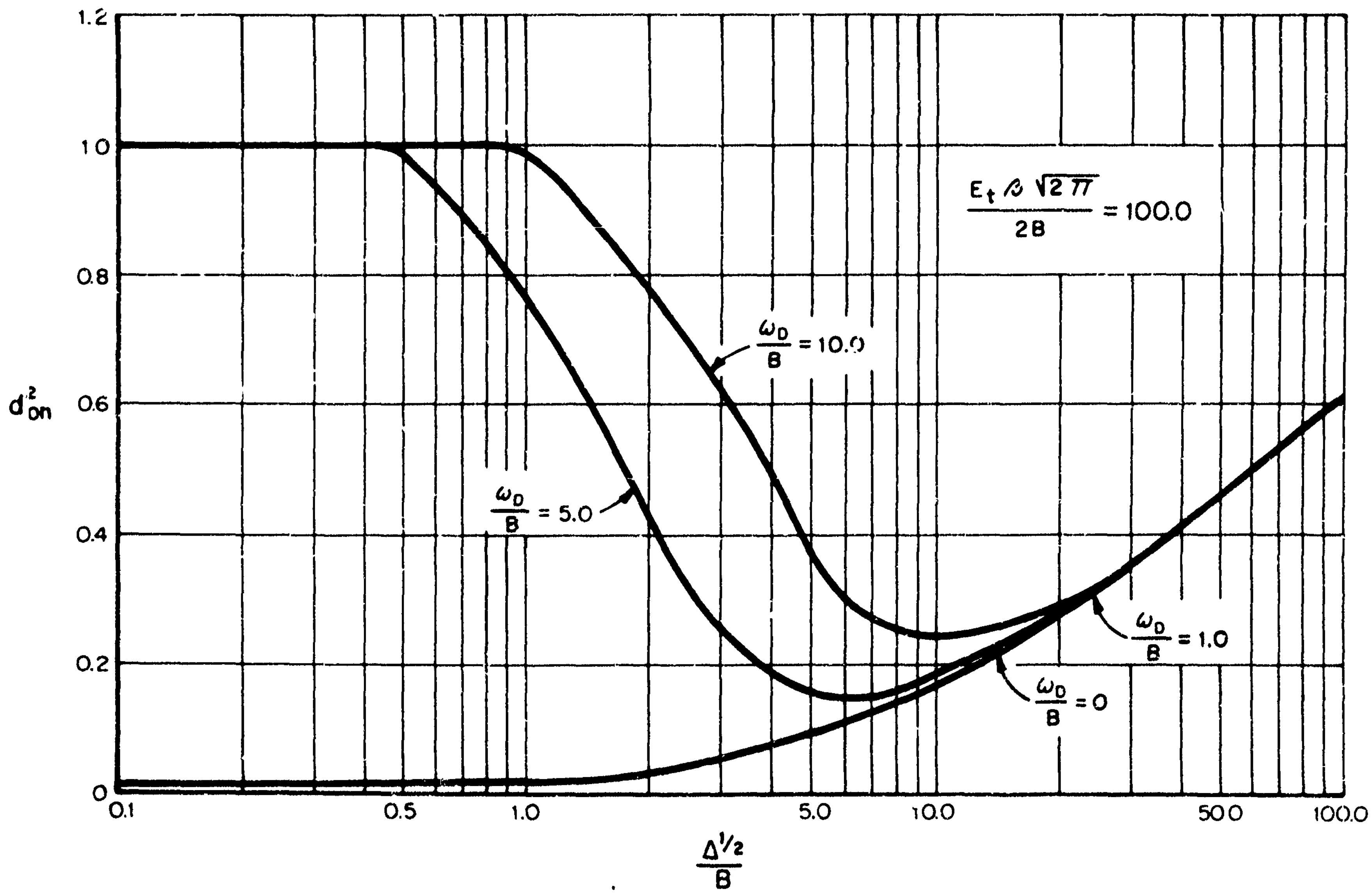

FIGURE 19 OPTIMUM RECEIVER PERFORMANCE (D= 100.0)

One's first reaction is that since the difference is about 1.5 db, the value of an optimum filter is questionable. However, this comparison is somewhat misleading. Let us assume that we require a given d^2 to obtain a desired performance level. Denote this required d^2 by d_r^2. Assume that, with the above parameters, the optimum system provides the required d_r^2. Thus,

$$d_{no}^2 \cdot \frac{2E_r}{N_o} = d_r^2 \tag{VI-52}$$

Now we want to find how much we must increase the transmitted signal energy to achieve the same d_r^2 with conventional processing. If we increase the transmitted energy by a factor k, we have:

$$E_{rc} = k\, E_{ro} \tag{VI-53}$$

and

$$E_{tc} = k\, E_{to} \tag{VI-54}$$

From Equations VI-34 and VI-48, we have:

$$d_{no}^2 \cdot \frac{2E_{ro}}{N_o} = \frac{\frac{2E_{ro}}{N_o} \cdot k}{1 + Dk \frac{1}{\left(1 + 2\frac{\Delta}{B^2}\right)^{\frac{1}{2}}} \exp - \frac{1}{2}\left\{\frac{\left(\omega_D/B\right)^2}{\left(1 + 2\frac{\Delta}{B^2}\right)}\right\}} \tag{VI-55}$$

For the above case:

$$k = \frac{d_{no}^2}{1 - d_{no}^2 \cdot D \frac{1}{\left(1 + 2\frac{\Delta}{B^2}\right)^{\frac{1}{2}}} \exp - \frac{1}{2}\left\{\frac{\left(\omega_D/B\right)^2}{\left(1 + 2\,\Delta/B^2\right)}\right\}} \tag{VI-56}$$

(If it is impossible to achieve the required d_r^2 by increasing the transmitted energy, Equation VI-54 may have a negative solution. Clearly, k must be positive to be meaningful.)

Evaluating, we obtain:

$$k = 2.45 \ (+3.9 \text{ db}) \qquad \text{(VI-57)}$$

This increase in energy required when a conventional receiver is used instead of an optimum receiver gives a more accurate picture of the cost of non-optimum processing. The difference in the two results occurs because as the transmitted energy increases, the reverberation return also increases. Thus, for high reverberation levels, one cannot combat the reverberation by raising the energy level.

The conclusions to be drawn from this section are two-fold:

a. The most important step in combating reverberation is proper signal design. This signal design can be accomplished using a conventional receiver. As pointed out in Section V, one will not achieve a unique maximum. The allowable range of signal parameters will be governed by such factors as available bandwidth, pulse duration, sound path stability, and other system constraints. The choice of signal will also be governed by what range of target velocities is of interest.

b. The second step is to consider how much one would gain by using an optimum receiver instead of a conventional receiver. One must examine the specific situation to see if the added complexity is warranted.

Now we want to conduct a similar analysis for the non-stationary case.

VII. NON-UNIFORM REVERBERATION

In this section, we consider the performance of conventional and optimum receivers in the presence of non-uniform reverberation. As discussed in Section III, if the distribution in range of the scatterers is non-uniform, then the reverberation return will be a sample function from a non-stationary Gaussian process. We consider first the conventional filter performance. In this section, we restrict ourselves to pulses with Gaussian envelopes and linear FM.

A. CONVENTIONAL FILTER

As pointed out in Section IV, to evaluate the conventional receiver, we require the ambiguity function of the signal and the scattering of the reverberation. First, evaluate the ambiguity function.

From Equation III-24, we have:

$$\theta(\tau, \omega) = \int_{-\infty}^{+\infty} \tilde{f}\left(t - \frac{\tau}{2}\right) \tilde{f}^*\left(+ \frac{\tau}{2}\right) e^{-j\omega t}\, dt \tag{VII-1}$$

For a Gaussian pulse,

$$\sqrt{2E_t}\, \tilde{f}(t) = \left(\frac{8aE_t^2}{\pi}\right)^{\frac{1}{4}} \exp\left\{ - (a + jb)\, t^2 \right\} \tag{VII-2}$$

First, let $b = 0$.

Then, substituting Equation VII-2 into VII-1, we have:

$$\theta(\tau, \omega) = \left(\frac{8aE_t^2}{\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{+\infty} \exp\left\{ - a\left(t - \frac{\tau}{2}\right)^2 - a\left(t + \frac{\tau}{2}\right)^2 - j\omega t \right\} dt \tag{VII-3}$$

Completing the square and integrating:

$$\theta(\tau,\omega) = 2E_t \exp\left\{-\frac{a\tau^2}{2} - \frac{\omega^2}{8a}\right\} \tag{VII-4}$$

Then,

$$\Psi(\tau,\omega) = \left|\theta(\tau,\omega)\right|^2 = 4E_t^2 \exp\left\{-a\tau^2 - \frac{\omega^2}{4a}\right\} \tag{VII-5}$$

Normalizing,

$$\Psi_n(\tau,\omega) \triangleq \frac{\Psi(\tau,\omega)}{\Psi(o,o)} = \exp\left\{-a\tau^2 - \frac{\omega^2}{4a}\right\} \tag{VII-6}$$

A sketch of the ambiguity function is shown in Figure 20.

For $b \neq 0$, we obtain:/

$$\theta(\tau,\omega) = 4E_t^2 \exp\left\{-\frac{a\tau^2}{2} - \frac{(\omega - 2b\tau)^2}{8a}\right\} \tag{VII-7}$$

and

$$\Psi_n(\tau,\omega) = \exp\left\{-a\tau^2 - \frac{(\omega - 2b\tau)^2}{4a}\right\} \tag{VII-8}$$

The equal amplitude lines for the linear FM cases are shown in Figure 21.

/ One can either evaluate the integral or use Theorem 4, Siebert (Reference 17).

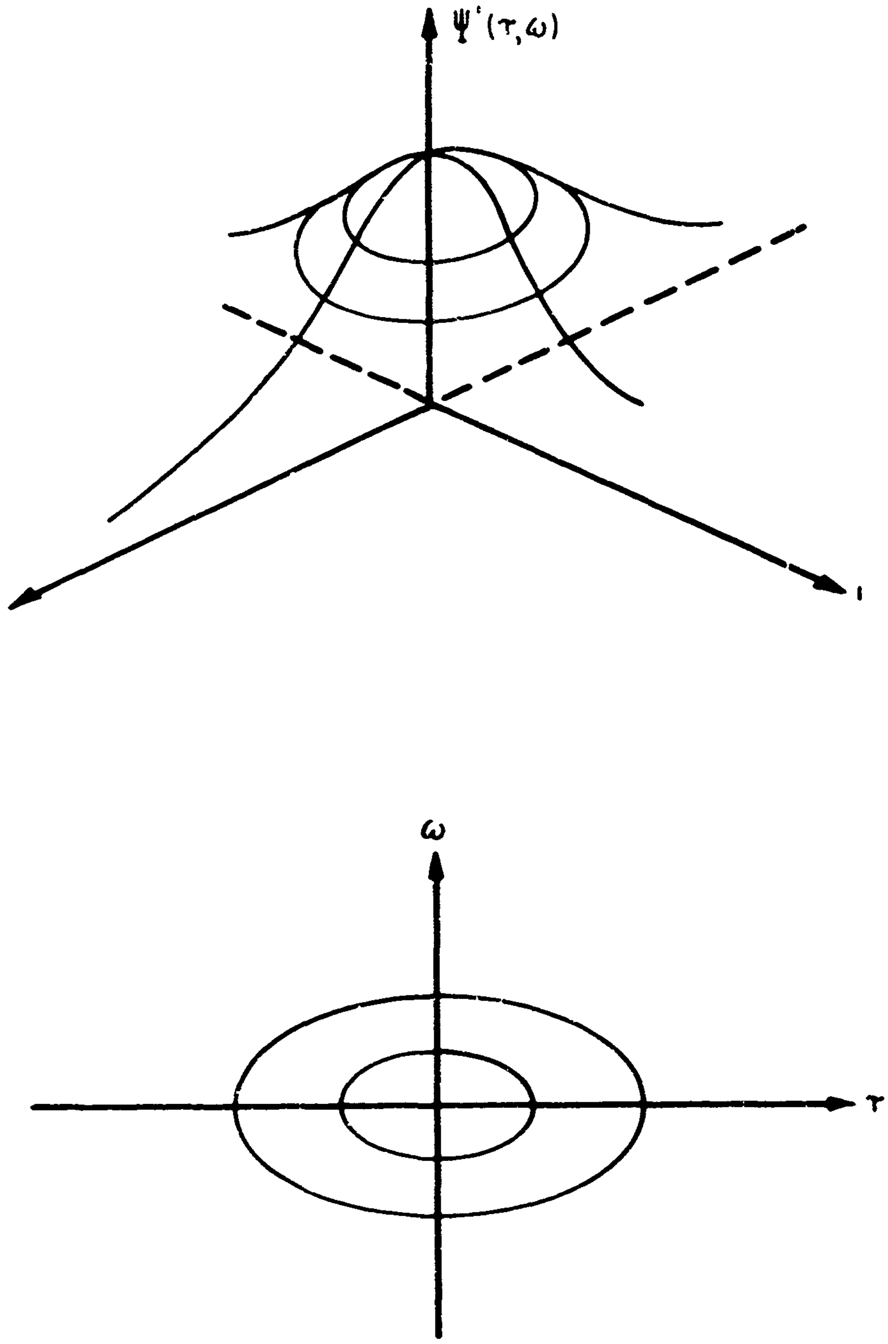

FIGURE 20 AMBIGUITY FUNCTION, GAUSSIAN PULSE (b = 0)

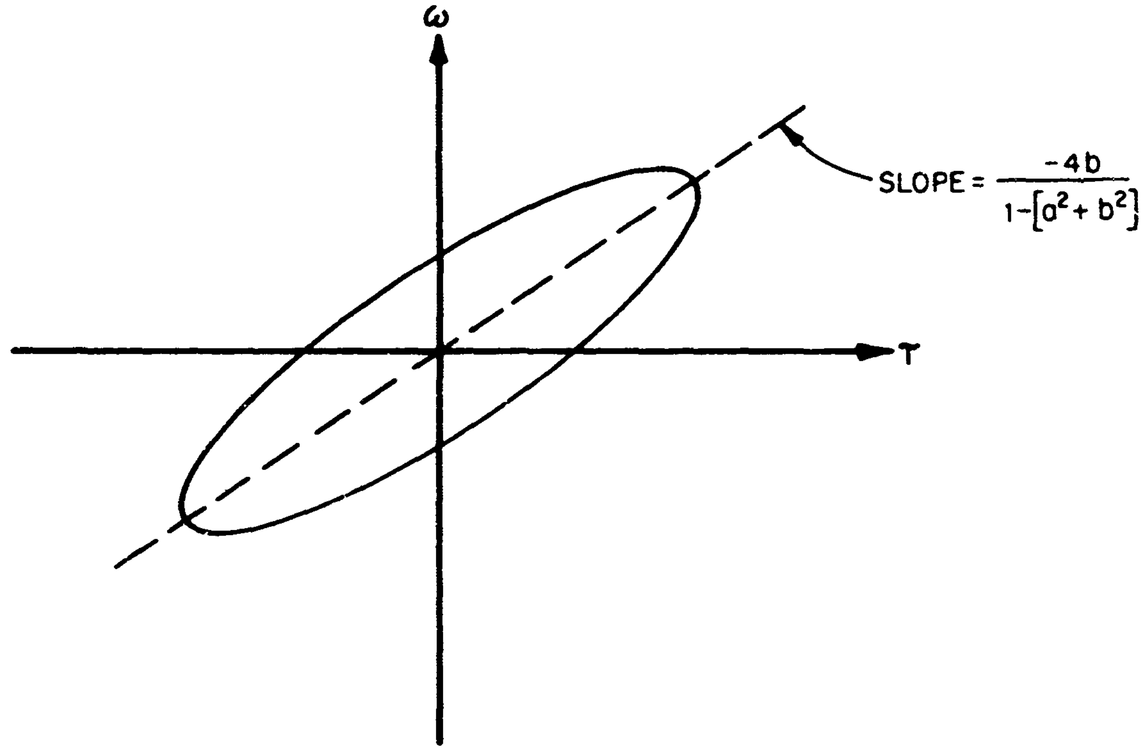

FIGURE 21 AMBIGUITY FUNCTION, LINEAR FM PULSE

We assume that the scattering density is a skewed Gaussian density.

$$s(x, Z) = \frac{1}{2\pi\sqrt{1-\rho^2}\,LB} \exp - \frac{1}{2} \frac{B^2x^2 - 2BL\rho\, xZ + L^2Z^2}{B^2L^2(1-\rho^2)} \tag{VII-9}$$

The simplest case is $\rho = 0$ (no skew).

Then,

$$s(x, Z) = \frac{1}{2\pi LB} \exp \left\{ -\frac{1}{2}\left(\frac{x^2}{L^2} + \frac{Z^2}{B^2}\right)\right\} \tag{VII-10}$$

Substituting Equations VII-6 and VII-10 into Equation IV-84, we have:

$$d_c^2 = \frac{\dfrac{2E_r}{N_o}}{\left\{1 + \dfrac{\frac{1}{2}E\{|Z|^2\}E_t}{N_o} \displaystyle\iint dx\, d\omega\, s(x,\omega)\, \Psi(\tau_d - x; \omega_d - \omega)\right\}}$$

$$d_c^2 = \frac{2E_r}{N_o}\left\{1 + \frac{\frac{1}{2}E\{|Z|^2\}E_t}{N_o} \iint d\tau\, d\omega \cdot \frac{1}{2\pi LB} \exp\left\{-\frac{1}{2}\left(\frac{\tau^2}{L^2} + \frac{\omega^2}{B^2}\right)\right\} \exp\left\{-a(\tau_d - \tau)^2 - \frac{(\omega_d - \omega)^2}{4a}\right\}\right\}^{-1} \tag{VII-11}$$

Evaluating, we have:

$$d_c^2(\tau_d, \omega_d) = \frac{2E_r}{N_o} \left\{ 1 + \frac{\frac{1}{2} E\{|Z|^2\} E_t}{N_o} \; \frac{1}{\left(L^2 + \frac{1}{2a}\right)^{\frac{1}{2}}} \cdot \frac{1}{\left(B^2 + 2a\right)^{\frac{1}{2}}} \right.$$

$$\left. \exp\left\{ -\frac{1}{2} \left[\frac{\tau_d^2}{L^2 + \frac{1}{2a}} + \frac{\omega_d^2}{B^2 + 2a} \right] \right\} \right\}^{-1} \qquad \text{(VII-12)}$$

First look at the behavior of d^2_{conv} (0, 0) as a function of a.

The worst case is when

$$a = \frac{B}{2L} \triangleq a_\omega \qquad \text{(VII-13)}$$

Then,

$$d_c^2(0,0) = \frac{2E_r}{N_o} \left\{ 1 + \frac{\frac{1}{2} E\{|Z|^2\} E_t}{N_o} \; \frac{1}{BL + 1} \right\}^{-1} \qquad \text{(VII-14)}$$

A convenient way to examine the behavior of d^2_{conv} (0, 0) for other a is to define:

$$a \triangleq k \frac{B}{2BL} = k\, a_\omega \qquad \text{(VII-15)}$$

and examine $\ln \dfrac{d^2_{conv}(0,0)}{2E_r/N_o} \triangleq \ln d^2_{nc}(0,0)$ (VII-16)

Thus,

$$-\ln d_{nc}^{2}(0,0) = +\ln\left\{1 + \frac{\frac{1}{2}E\{|Z|^{2}\}E_{t}}{N_{o}} \quad \frac{1}{(1+kBL)^{\frac{1}{2}}\left(1+\frac{BL}{k}\right)^{\frac{1}{2}}}\right\} \tag{VII-17}$$

In Figure 22, we plot $(1 + kBL)^{\frac{1}{2}} \left(1 + \frac{BL}{k}\right)^{\frac{1}{2}}$ as a function of k for $BL = 1$. Since the function is symmetric (on a ln scale) about $k = 1$, we need to plot only $k \geq 1$. Observe that the nearer to one the value of this function, the better the system performance.

In Figures 23, 24, 25, and 26,/ we plot $d_{nc}^{2}(0,0)$ for various values of $\frac{\frac{1}{2}E\{|Z|^{2}\}E_{t}}{N_{o}}$, BL, and k.

For non-zero τ_D and ω_D, we want to choose a to make the term

$$\frac{\frac{1}{2}E\{|Z|^{2}\}E_{t}}{N_{o}} \frac{1}{\left(L^{2}+\frac{1}{2a}\right)^{\frac{1}{2}}\left(B^{2}+2a\right)^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}\left[\frac{\tau_{D}^{2}}{L^{2}+\frac{1}{2a}} + \frac{\omega_{D}^{2}}{B^{2}+2a}\right]\right\}$$

as small as possible.

As before, there is no unique minimum. There is a worst value of a. This worst value is a function of B, L, ω_D, and τ_D. There does not seem to be a simple analytic expression for the worst value.

Similarly one can use the expression in Equation VII-8 for the ambiguity function of a Gaussian pulse with linear FM.

/ The optimum filter result is also plotted. This will be derived in Section VII-B and discussed at that time.

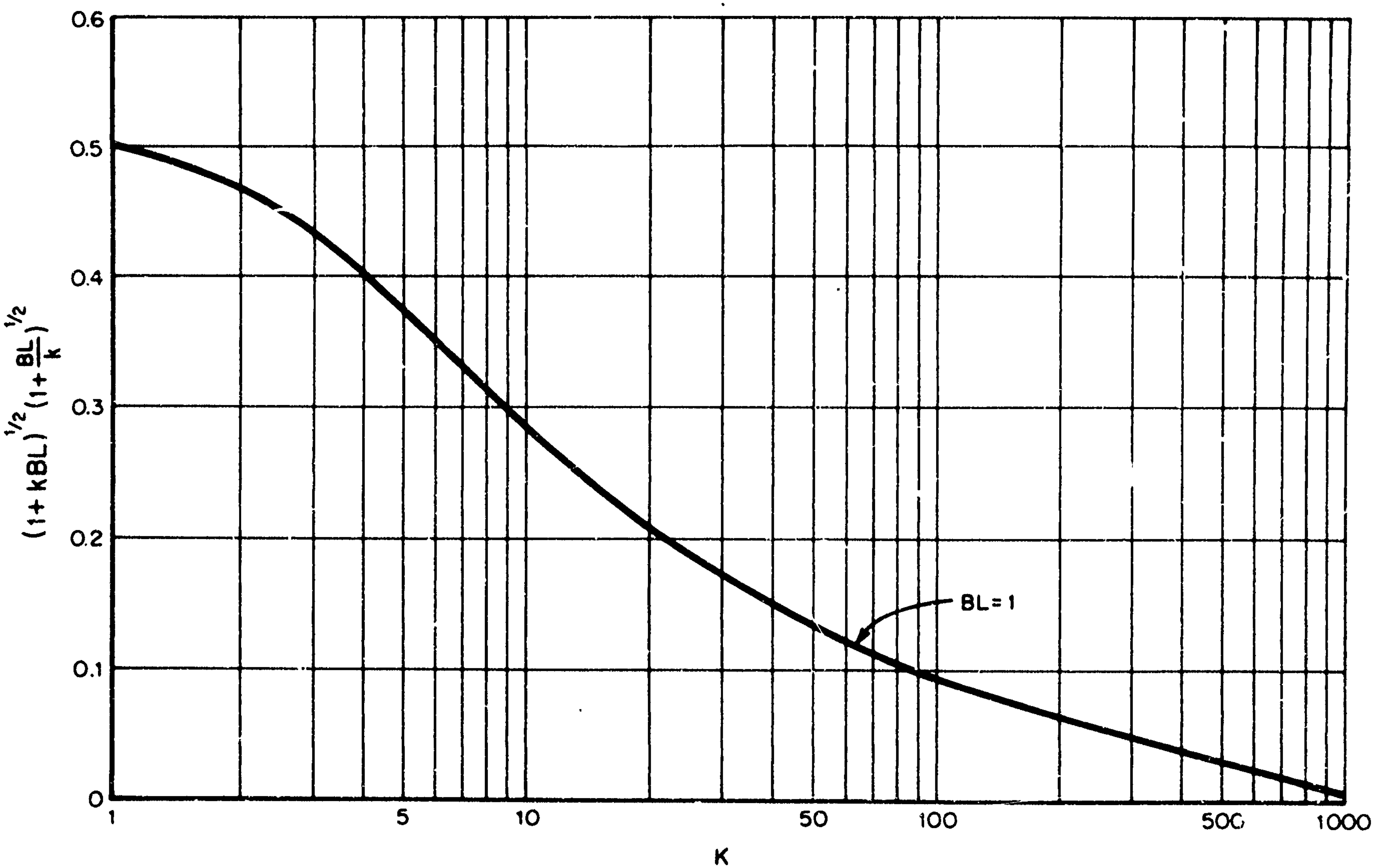

FIGURE 22 DEGRADATION AS A FUNCTION OF K

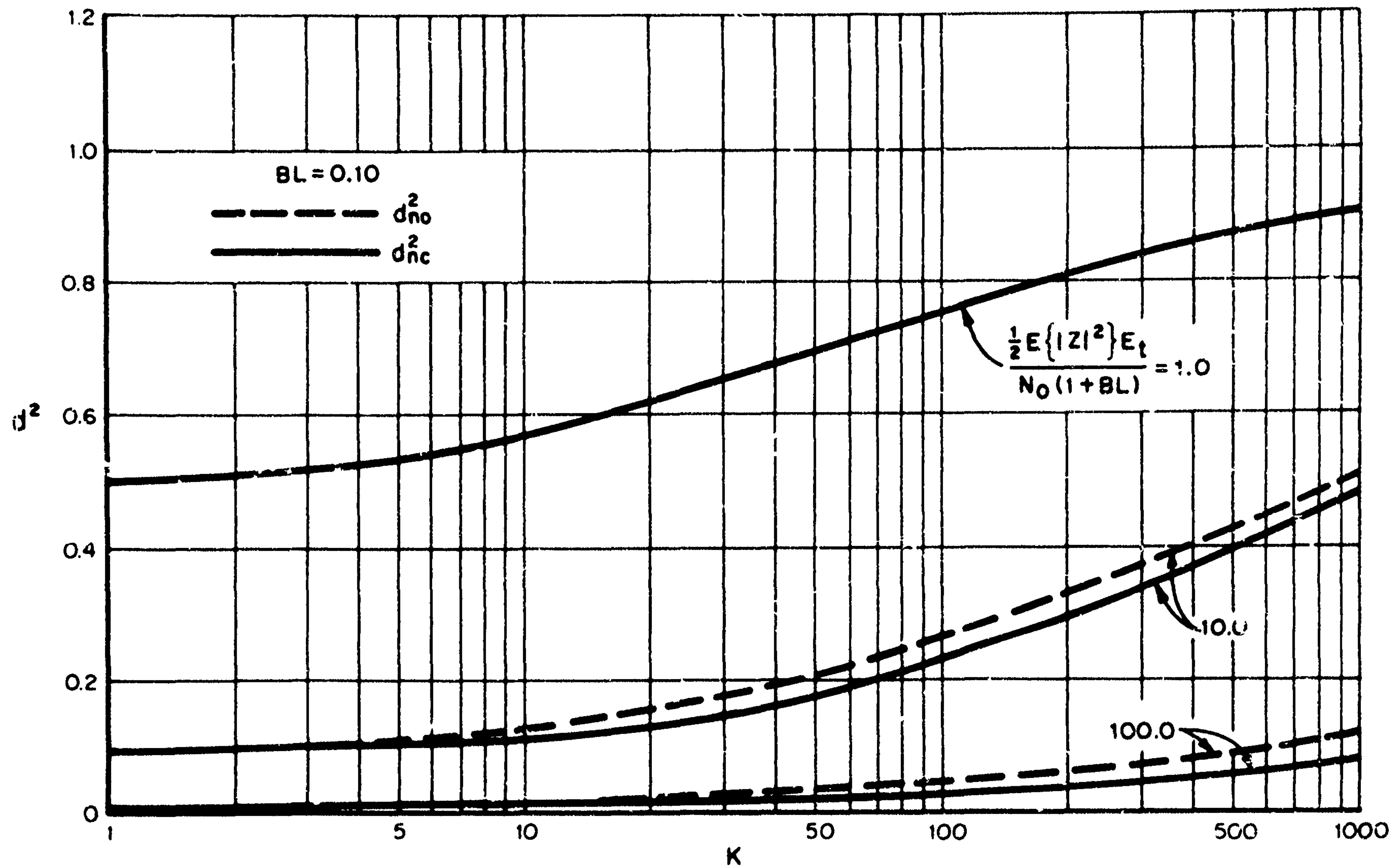

FIGURE [illegible]3 PERFORMANCE OF CONVENTIONAL AND OPTIMUM RECEIVERS -- BL = 0.10

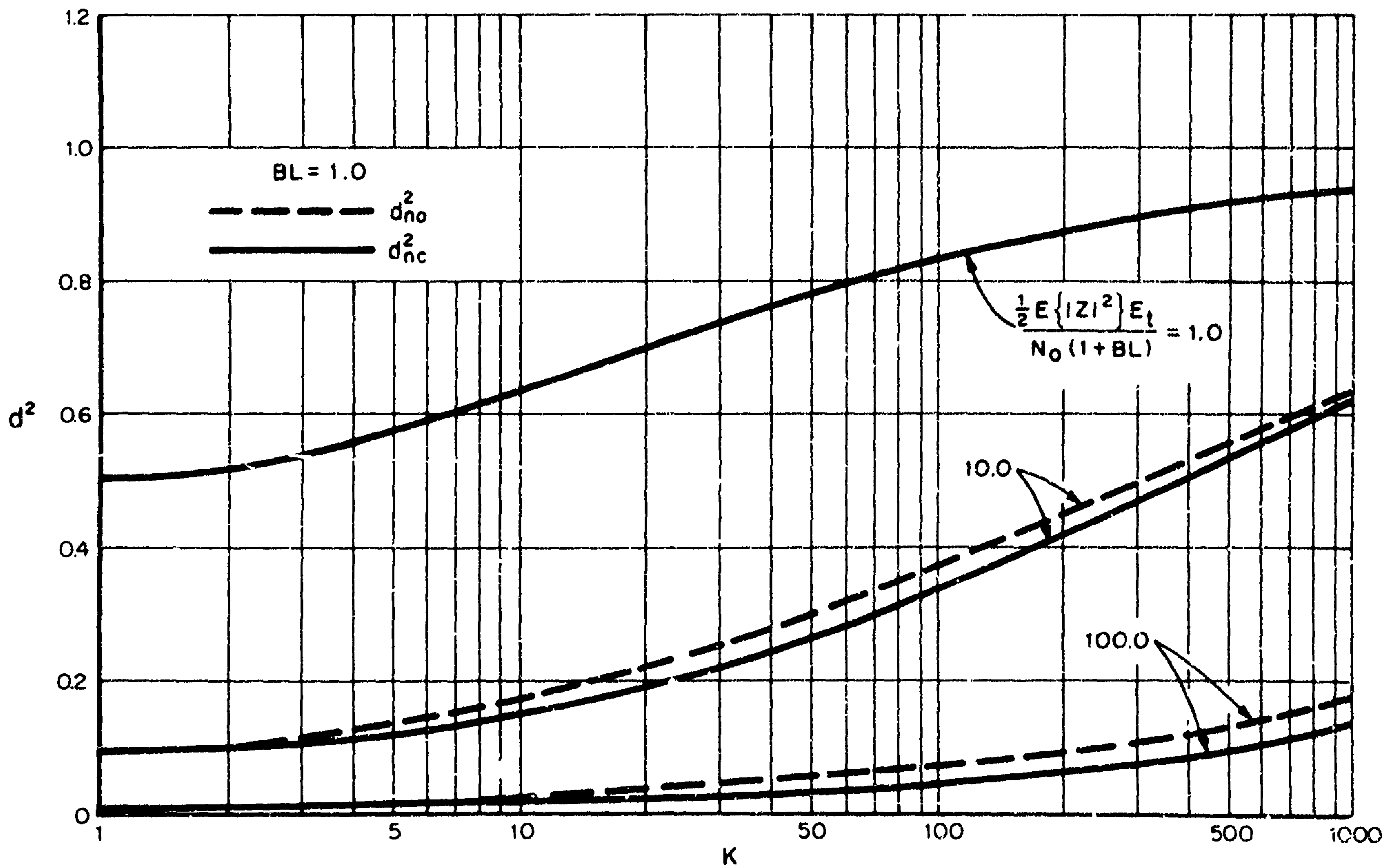

FIGURE 24 PERFORMANCE OF CONVENTIONAL AND OPTIMUM RECEIVERS -- BL = 1.0

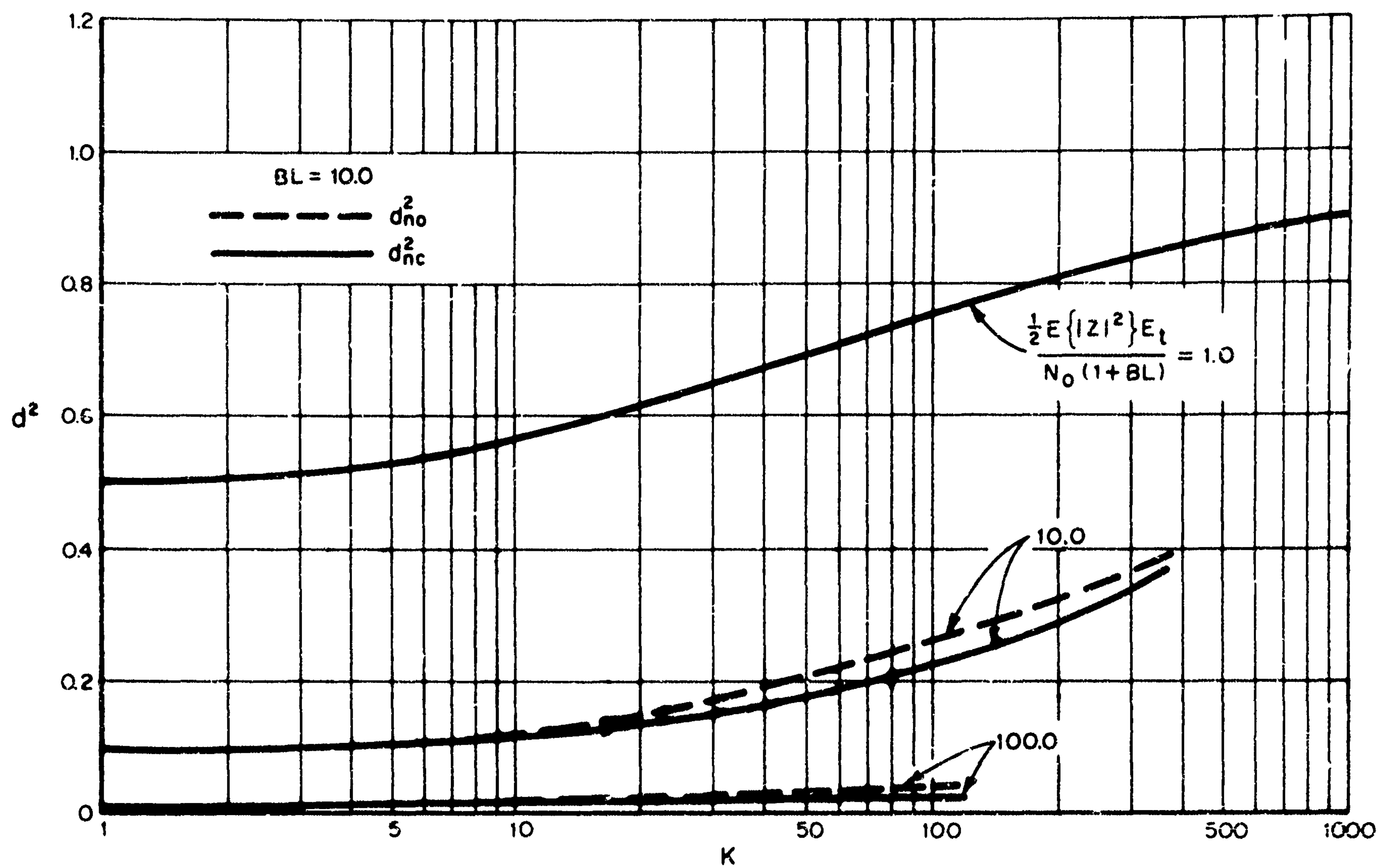

FIGURE 25 PERFORMANCE OF CONVENTIONAL AND OPTIMUM RECEIVERS -- BL = 10.0

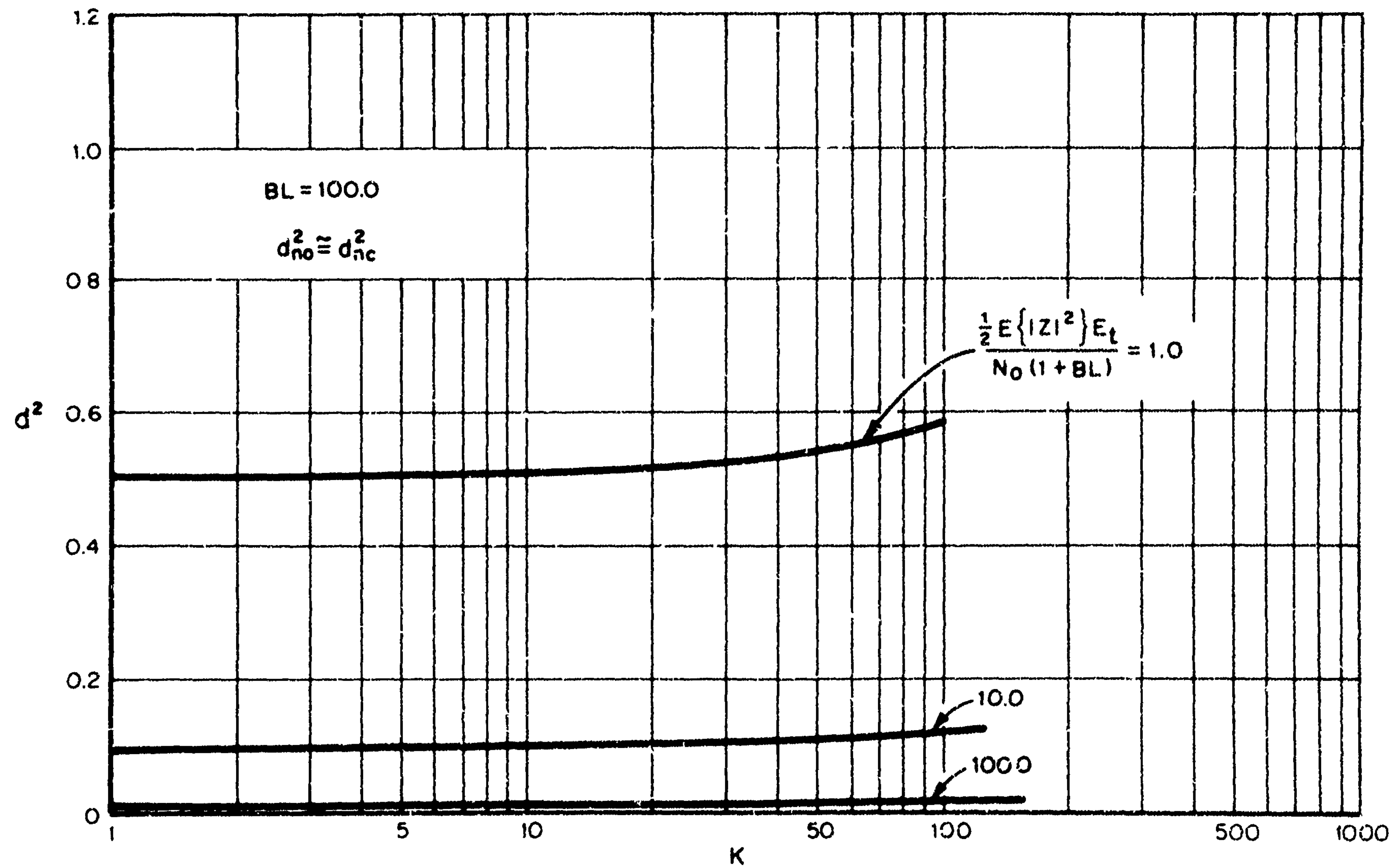

FIGURE 26 PERFORMANCE OF CONVENTIONAL AND OPTIMUM RECEIVERS -- BL = 100.0

B. OPTIMUM FILTER

To find the optimum filter, we must solve the integral equation derived in Section IV:

$$\tilde{s}_d(t_\alpha) = \int_{-\infty}^{\infty} \tilde{R}_r(t_\alpha, t_\beta)\, \tilde{q}(t_\beta)\, dt_\beta \tag{VII-18}$$

where

$$\tilde{R}_r(t_\alpha, t_\beta) = N_o\, \delta(t_\alpha - t_\beta) + \tilde{R}_{n_r}(t_\alpha, t_\beta) \tag{VII-19}$$

and

$$\tilde{R}_{n_r}(t_\alpha, t_\beta) = \frac{1}{2}\, \frac{\frac{1}{2} E\{|Z|^2\}}{2\pi} \int_{-\infty}^{\infty} S(-\omega;\, t_\alpha - t_\beta)\, \theta(t_\beta - t_\alpha; \omega)\, e^{+j\omega\left(\frac{t_\alpha + t_\beta}{2}\right)} dw \tag{VII-20}$$

$$\begin{aligned}\tilde{R}_{n_r}(t_\alpha, t_\beta) = \frac{1}{2}\, \frac{\frac{1}{2} E\{|Z|^2\}}{2\sqrt{2\pi}} \cdot \frac{2E_t}{\left(L^2 + \frac{1}{4a}\right)^{\frac{1}{2}}} \exp - \frac{1}{2} \Bigg\{ & t_\alpha^2 \left[(B^2 + a) + \frac{1}{4\left(L^2 + \frac{1}{4a}\right)} \right] \\ & + t_\beta^2 \left[(B^2 + a) + \frac{1}{4\left(L^2 + \frac{1}{4a}\right)} \right] \\ & - 2t_\alpha t_\beta \left[(B^2 + a) - \frac{1}{4\left(L^2 + \frac{1}{4a}\right)} \right] \Bigg\}\end{aligned} \tag{VII-21}$$

To solve the integral equation, we will expand the kernel in a suitable bi-orthonormal expansion.

A suitable series can be obtained from Mehler's expansion,/ which is:

$$\frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{x^2+y^2-2\rho xy}{2(1-\rho^2)}\right\} = \frac{1}{2\pi}\left[\exp -\frac{x^2}{2} - \frac{y^2}{2}\right] \sum_0^\infty \rho^n \frac{H_n(x)}{(n!)^{\frac{1}{2}}} \cdot \frac{H_n(y)}{(n!)^{\frac{1}{2}}} \qquad \text{(VII-22)}$$

$$|\rho| < 1$$

where $H_n(x)$ is the n^{th} Hermite function.

Since

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H_m(x)\, H_n(x) \exp\left\{-\frac{x^2}{2}\right\} dx = \begin{cases} n! & n = m \\ 0 & \text{otherwise} \end{cases} \qquad \text{(VII-23)}$$

A suitable set of orthonormal functions are:

$$\phi_n(x) = \frac{1}{(2\pi)^{\frac{1}{4}} (n!)^{\frac{1}{2}}} H_n(x) \exp\left\{-\frac{x^2}{4}\right\} \qquad \text{(VII-24)}$$

From Equation VII-22, we obtain the desired expansion:

$$\frac{1}{\sqrt{2\pi}\,(1-\rho^2)^{\frac{1}{2}}} \exp -\frac{x^2(1+\rho^2)+y^2(1+\rho^2)-4\rho xy}{4(1-\rho^2)} = \sum_{n=0}^{\infty} \rho^n \phi_n(x)\, \phi_n(y) \qquad \text{(VII-25)}$$

We observe that all of the eigenvalues of the Gaussian kernel are specified by a single number ρ.

/ See Reference 18. Our approach here is related to that in Reference 19.

Now we want to expand $\widetilde{R}_{n_r}(t_\alpha, t_\beta)$ as given by Equation VII-21 into a series of the form of Equation VII-25.

Introduce two new variables:

$$C^2 = B^2 + a \qquad \text{(VII-26)}$$

$$D^2 = \frac{1}{4\left(L^2 + \frac{1}{4a}\right)} \qquad \text{(VII-27)}$$

Then,

$$\widetilde{R}_{n_r}(t_\alpha, t_\beta) = \frac{1}{2} \frac{\frac{1}{2} E\{|Z|^2\}}{\sqrt{2\pi}} 4DE_t \exp - \frac{1}{2} \left\{ t_\alpha^2 (C^2 + D^2) + t_\beta^2 (C^2 + D^2) - 2t_\alpha t_\beta (C^2 - D^2) \right\} \qquad \text{(VII-28)}$$

Now let $x = t\sigma$, $y = t_\beta \sigma$ and solve for σ and ρ such that Equation VII-28 is identical to Equation VII-22.

We obtain

$$\rho = \frac{C - D}{C + D} \qquad \text{(VII-29)}$$

$$\sigma^2 = \frac{1}{4CD} \qquad \text{(VII-30)}$$

After a little algebra, we obtain:

$$\widetilde{R}_{n_r}(t_\alpha, t_\beta) = \frac{\frac{1}{2} E\{|Z|^2\}}{2} \left[\frac{8aE_t^2}{\pi(1 + 4aL^2)} \right]^{\frac{1}{2}} \frac{\sqrt{2\pi}}{(C+D)^{\frac{1}{2}}} \sum_{k=0}^{\infty} \left[\frac{(C-D)}{(C+D)} \right]^k \Psi_k(t_\alpha) \Psi_k(t_\beta) \qquad \text{(VII-31)}$$

where

$$\Psi_k(t_\alpha) = \frac{H_n\left[t_\alpha\,[4CD]^{\frac{1}{2}}\right]\exp\left\{-t_\alpha^2 CD\right\}}{(n!)^{\frac{1}{2}}\,(2\pi)^{\frac{1}{4}}\left(\frac{1}{4CD}\right)^{\frac{1}{4}}} \qquad \text{(VII-32)}$$

Including the white noise term, we obtain from Equations VII-19 and VII-31:

$$\widetilde{R}_r(t_\alpha, t_\beta) = N_o\,\delta(t_\alpha - t_\beta) + K_r \sum_{k=0}^{\infty} \rho^k\,\Psi_k(t_\alpha)\,\Psi_k(t_\beta) \qquad \text{(VII-33)}$$

where

$$K_r = \frac{\frac{1}{2}\,E\{|Z|^2\}}{2}\left[\frac{8aE_t^2}{\pi(1+4aL^2)}\right]^{\frac{1}{2}}\frac{\sqrt{2\pi}}{(C+D)^{\frac{1}{2}}} \qquad \text{(VII-34)}$$

Now we define an inverse to $\widetilde{R}_r(t_\alpha, t_\beta)$.

$$\int_{-\infty}^{\infty} \widetilde{R}_r(t_\alpha, t_\beta)\,\widetilde{r}_r(t_\beta, t_\gamma)\,dt_\beta = \delta(t_\alpha - t_\gamma) \qquad \text{(VII-35)}$$

It is easy to verify that:

$$\widetilde{r}_r(t_\alpha, t_\beta) = \frac{1}{N_o}\,\delta(t_\alpha - t_\beta) - \frac{1}{N_o}\sum_{n=1}^{\infty}\frac{K_r\rho^n}{N_o + K_r\rho^n}\,\Psi_n(t_\alpha)\,\Psi_n(t_\beta) \qquad \text{(VII-36)}$$

Let C_n denote the coefficient in the sum:

$$C_n \triangleq \frac{K_r \rho^n}{N_o + K_r \rho^n} = \frac{\rho^n}{\rho^n + \frac{N_o}{K_r}} \tag{VII-37}$$

Multiply both sides of Equation VII-18 by $\tilde{r}_r(t_\alpha, t_\beta)$ and integrate. This gives:

$$\int_{-\infty}^{\infty} \tilde{S}_d(t_\alpha)\, \tilde{r}_r(t_\alpha, t_\beta)\, dt_\alpha = \iint_{-\infty}^{\infty} \tilde{r}_r(t_\alpha, t_\beta)\, \tilde{R}_r(t_\alpha, t_\gamma)\, \tilde{q}(t_\gamma)\, dt_\alpha\, dt_\gamma \tag{VII-38}$$

which reduces to:

$$\tilde{q}(t_\alpha) = \int_{-\infty}^{\infty} \tilde{S}_d(t_\alpha)\, \tilde{r}_r(t_\alpha, t_\beta)\, dt_\beta \tag{VII-39}$$

Substituting Equation VII-36 into VII-39, we obtain:

$$\tilde{q}(t_\alpha) = \frac{1}{N_o} \tilde{S}_d(t_\alpha) - \frac{1}{N_o} \sum_{n=0}^{\infty} C_n \psi_n(t_\alpha) \int_{-\infty}^{\infty} \psi_n(t_\beta)\, S_d(t_\beta)\, dt_\beta \tag{VII-40}$$

Let

$$r_n \triangleq \int_{-\infty}^{\infty} \psi_n(t_\beta)\, S_d(t_\beta)\, dt_\beta \tag{VII-41}$$

Then,

$$\tilde{q}(t_\alpha) = \frac{1}{N_o}\tilde{S}_d(t_\alpha) - \frac{1}{N_o}\sum_{n=1}^{\infty} C_n r_n \Psi_n(t_\alpha) \tag{VII-42}$$

which specifies the optimum detector. It is simply a band pass filter whose complex impulse response is matched to $\tilde{q}(t)$.

To evaluate the performance, we recall from Equation IV-44 that:

$$d_o^2 = \int_{-\infty}^{\infty} \tilde{S}_d(t_\alpha)\, q^*(t_\alpha) \tag{VII-43}$$

Therefore, using Equation VII-42, we obtain:

$$d_o^2 = \frac{1}{N_o}\left\{\int_{-\infty}^{\infty}\tilde{S}_d(t_\alpha)\,\tilde{S}_d^*(t_\alpha)\,dt_\alpha - \sum_{n=1}^{\infty} C_n r_n \int_{-\infty}^{\infty}\Psi_n(t_\alpha)\, S_d(t_\alpha)\, dt_\alpha\right\} \tag{VII-44}$$

which reduces to:

$$d_o^2 = \frac{2E_r}{N_o} - \frac{1}{N_o}\sum_{n=1}^{\infty} C_n \left|r_n\right|^2 \tag{VII-45}$$

Now, the $\Psi_n(t)$ are a CON set, so we may write:

$$\tilde{S}_d(t_\alpha) = \sum_{n=0}^{\infty} r_n^* \Psi_n(t_\alpha) \tag{VII-46}$$

which implies:

$$\int_{-\infty}^{\infty} \tilde{S}_d(t)\ \breve{S}_d^*(t_\alpha)\ dt_\alpha = \sum_{n=0}^{\infty} |r_n|^2 = 2E_r \qquad \text{(VII-47)}$$

Since the eigenvalues are monotones decreasing with increasing n, it follows from Property 1 of Section V that d^2_{opt} is a minimum when

$$r_n = 0 \qquad n \neq 0 \qquad \text{(VII-48)}$$

or

$$\breve{S}_d(t_\alpha) = r_o\ \psi_o(t_\alpha) \qquad \text{(VII-49)}$$

We see that this implies:

$$\exp\left\{-a_w t^2\right\} = \exp\left\{-\frac{\left[16a_w(B^2 + a_w)\right]^{\frac{1}{2}}}{\left[1 + 4a_w L^2\right]^{\frac{1}{2}}}\ t_2\right\} \qquad \text{(VII-50)}$$

or

$$a_w = \frac{B}{2L} \qquad \text{(VII-51)}$$

From Properties 2 and 3 of Section V, we know that Equation VII-51 must be identical to Equation VII-13. In other words, the worst signal is identical for the conventional and optimum receiver.

To evaluate the error for the worst case, we must compute C_o and r_o.

From Equations VII-37 and VII-47, it is clear that:

$$r_o = \sqrt{2E_r} \qquad \text{(VII-52)}$$

$$C_o = \frac{\rho^o}{\rho^o + \frac{N_o}{K_r}} = \frac{1}{1 + \frac{N_o}{K_r}} \qquad \text{(VII-53)}$$

and from Equation VII-34

$$K_r = \frac{\frac{1}{2} E\{|Z|^2\} E_t}{BL+1} \qquad \text{(VII-54)}$$

Then,

$$d_{ow}^2 = \frac{2E_r}{N_o}\left[1 - C_o\right] \qquad \text{(VII-55)}$$

$$= \frac{\frac{2E_r}{N_o}}{1 + \frac{\Delta E_t}{N_o}\ \frac{1}{BL+1}} \qquad \text{(VII-56)}$$

Equation VII-56 gives the same result as Equation VII-14 since the filters are identical.

For the general case, the optimum filter and the conventional filter will be different. To evaluate the error, we must sum the series given by Equation VII-45.

Writing $a = ka_w$, we have:

$$K_r = \frac{1}{2} E\{|Z|^2\} E_t \ \frac{2k^{\frac{1}{2}}}{\left[(k + 2BL)^{\frac{1}{2}}\ (1 + k2BL)^{\frac{1}{2}} + k^{\frac{1}{2}}\right]^{\frac{1}{2}}} \qquad \text{(VII-57)}$$

and

$$\rho = \frac{(k + 2BL)^{\frac{1}{2}} (1 + k2BL)^{\frac{1}{2}} - k^{\frac{1}{2}}}{(K + 2BL)^{\frac{1}{2}} (1 + k2BL)^{\frac{1}{2}} + k^{\frac{1}{2}}} \tag{VII-58}$$

and

$$r_n = \int_{-\infty}^{\infty} \psi_n(t)\, \tilde{S}_d(t)\, dt \tag{VII-59}$$

Now,

$$\tilde{S}_d(t) = \left(\frac{8aE_r^2}{\pi} \right)^{\frac{1}{4}} e^{-at^2} \tag{VII-60}$$

and

$$\Psi_n(t) = \frac{H_n \left[t \left[4CD \right]^{\frac{1}{2}} \right] e^{-t^2 CD}}{(n!)^{\frac{1}{2}} (2\pi)^{\frac{1}{4}} \left(\frac{1}{4CD} \right)^{\frac{1}{4}}} \tag{VII-61}$$

where

$$C^2 \triangleq \frac{B}{2L} \left[2BL + k \right] \tag{VII-62}$$

$$D^2 \triangleq \frac{B}{2L}\, r\, \frac{1}{(1 + 2BLr)} \tag{VII-63}$$

Then,

$$r_n = \frac{(16 E_r^2)^{\frac{1}{4}} a^{\frac{1}{4}}}{(n!)^{\frac{1}{2}} (2\pi)^{\frac{1}{2}} \left(\frac{1}{4CD}\right)^{\frac{1}{4}}} \int_{-\infty}^{\infty} H_n\left[t[4CD]^{\frac{1}{2}}\right] e^{-t^2 CD} e^{-at^2} dt \qquad \text{(VII-64)}$$

Letting $x = t[4CD]^{\frac{1}{2}}$, this reduces to:

$$r_n = \frac{(2E_r)^{\frac{1}{2}} (2)^{\frac{1}{2}} a^{\frac{1}{4}} (CD)^{\frac{1}{4}}}{(n!)^{\frac{1}{2}} (a + CD)^{\frac{1}{2}}} \left\{ \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} H_n(x) \exp\left(-\frac{x^2}{2\sigma^2}\right) dx \right\} \qquad \text{(VII-65)}$$

where

$$\sigma^2 \triangleq \frac{2CD}{a + CD} \qquad \text{(VII-66)}$$

Denote the bracketed term by $G_n(\sigma)$:

$$G_n(\sigma) = \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} H_n(x) \exp\left(-\frac{x^2}{2\sigma^2}\right) dx \qquad \text{(VII-67)}$$

Substituting into Equation VII-45, we have:

$$d_o^2 = \frac{2E_r}{N_o} \left\{ 1 - \frac{2a^{\frac{1}{2}} (CD)^{\frac{1}{2}}}{(a + CD)} \sum_{n=0}^{\infty} \frac{G_n^2(\sigma)\, C_n}{n!} \right\} \qquad \text{(VII-68)}$$

Using the structure of the Hermite functions,/ we can find an expression for $G_n(\sigma)$.

/ See Cramer, Reference 20.

First, we evaluate $G_n(\sigma)$ for $n = 0, 2, 4, 6$ and deduce the expression for arbitrary n.

Hermite Functions	$G_n(\sigma)$	
$H_0(x) = 1$	$G_0(\sigma) = 1$	
$H_2(x) = x^2 - 1$	$G_2(\sigma) = \sigma^2 - 1$	
$H_4(x) = x^4 - 6x^2 + 3$	$G_4(\sigma) = 3(\sigma^2 - 1)^2$	
$H_6(x) = x^6 - 15x^4 + 45x^2 - 15$	$G_6(\sigma) = 15(\sigma^2 - 1)^3$	(VII-69)

Looking at the sequence, we see that

$$G_n(\sigma) = \frac{n!}{\left(\frac{n}{2}\right)! \, (2)^{\frac{n}{2}}} \left[\sigma^2 - 1\right]^{\frac{n}{2}} \qquad n \text{ even} \tag{VII-70}$$

$$= 0 \qquad n \text{ odd}$$

and

$$\sigma^2 - 1 = \frac{CD - a}{CD + a} \tag{VII-71}$$

Observe that for $\sigma = 1$,

$$G_0(1) = 1 \tag{VII-72}$$

$$G_n(1) = 0 \qquad n \neq 0$$

which corresponds to the case, $k = 1$.

Next, evaluate C_n.

$$C_n = \frac{\rho^n}{\rho^n + \frac{N_o}{K_r}} \tag{VII-73}$$

From Equations VII-57, VII-62, and VII-63 we have:

$$K_r = \frac{2\,\frac{1}{2}\,E\{|Z|^2\}\,E_t \cdot D}{C + D} \tag{VII-74}$$

and

$$\rho = \frac{C - D}{C + D} \tag{VII-75}$$

Substituting Equations VII-74 and VII-75 into Equations VII-73 and VII-68, we have:

$$C_n = \frac{2\,\frac{1}{2} E\{|Z|^2\}\,E_t}{N_o} \left(\frac{1}{\frac{2\,\frac{1}{2} E\{|Z|^2\}\,E_t}{N_o} + \frac{(C+D)^{n+1}}{D(C-D)^n}} \right) \tag{VII-76}$$

$$d_{no}^2 = \frac{d_o^2}{2E_r/N_o} = \left\{ 1 - \frac{2a^{\frac{1}{2}}(CD)^{\frac{1}{2}}}{(a + CD)} \sum_{m=0}^{\infty} \frac{(2m)!}{(m!2m)^2} (\rho^2 - 1)^{2m} C_{2m} \right\} \tag{VII-77}$$

where we have set $m = \frac{n}{2}$ since the odd terms in the series were zero.

We have evaluated the sum for the following parameter values:

1. $$\frac{\frac{1}{2} E\{|Z|^2\} E_t}{N_o} \quad \frac{1}{1+BL} = 1, 10, 100 \tag{VII-78}$$

2. BL = 0.1, 1, 10, and 100

The result d_{no}^2 is plotted as a function of k in Figures 23 - 26. When separate curves are not shown, d_{no}^2 is approximately equal to d_{nc}^2.

Also tabulated in Equation VII-14:

$$d_{nc}^2 \quad \frac{1}{1 + \frac{\frac{1}{2} E\{|Z|^2\} E_t}{N_o} \quad \frac{1}{1+BL}} \tag{VII-80}$$

From the tabulation, we observe that there is very little difference between d_{no}^2 and d_{nc}^2. There are several reasons for this result:

1. The most important reason is that we have considered only the case where the target has zero velocity. As shown in Figure 27, the target is exactly at the peak of the scattering function.

If the target had a non-zero velocity or were displaced in range from the peak of the scattering function, the difference between the optimum and conventional receiver would be larger. This is because the optimum filter uses its knowledge of the reverberation scattering function to partially "tune out" the reverberation. For the case we considered, the largest amount of reverberation was in the same range-Doppler location as the target. Thus, the optimum filter could not tune out the reverberation peak without also tuning out the target. For targets away from the peak, the optimum filter can use its knowledge more effectively.

2. A second reason is that the particular scattering function we have chosen is smooth in both directions. One of the advantages of a non-stationary model is that it uses the non-uniform distribution of scatterers along the path of sound wave to improve its detection capability. Intuitively, one would think that the more non-uniform the distribution is, the more useful knowledge of it would be. Thus, our choice of a smooth scattering function tends to negate the effect of an optimum filter.

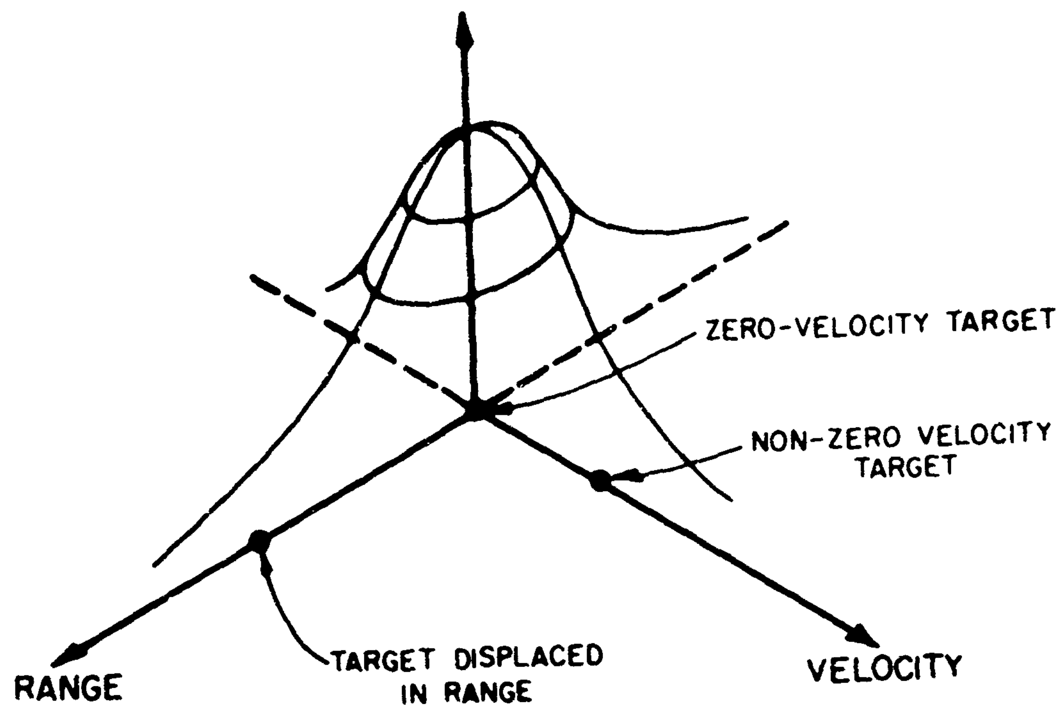

FIGURE 27 TARGET LOCATION WITH RESPECT TO REVERBERATION

REFERENCES

1. Woodward, P.M., Probability and Information Theory, with Applications to Radar, McGraw-Hill, New York; Pergamon, London.

2. Helstrom, C., Statistical Theory of Signal Detection, Pergamon Press, London, 1960.

3. Reed, I.S., "The Power Spectrum of Returned Echo from a Random Collection of Moving Scatterers," July 8, 1963, Technical Paper presented at IDA Summer Study.

4. Kelly, E.J., and E.C. Lerner, "A Mathematical Model for the Radar Echo from a Random Collection of Scatterers," Technical Report No. 123, M.I.T. Lincoln Laboratory, June 15, 1956.

5. Kelly, E.J., I.S. Reed, and W.L. Root, "The Detection of Radar Echoes in Noise, Part I," Jour. Soc. Indus. Appl. Math., 8, 2, June 1960.

6. Parzen, E., Stochastic Processes, Holden-Day, Inc., San Francisco, California, 1962.

7. Davenport, W.B., and W.L. Root, An Introduction to the Theory of Random Signals and Noise, McGraw-Hill, New York, 1958.

8. Rice, S. O., "Mathematical Analysis of Random Noise," Bell System Technical Journal, 23, 282-332, 1944.

9. Siebert, W.M., "A Radar Detection Philosophy," Trans. I.R.E., IT-2, 204, Sept. 1956.

10. Peterson, W.W., T.G. Birdsall, and W.C. Fox, "The Theory of Signal Detectability," Trans. I.R.E., PGIT-4, 171, Sept. 1954.

11. Marcum, J.I., and P. Swerling, "Studies of Target Detection by Pulsed Radar," Trans. I.R.E. on Information Theory, IT-6, 59-308, April 1960.

12. Marcum, J.I., "A Statistical Theory of Target Detection by Pulsed Radar: Mathematical Appendix," Rand Corporation Report RM-753, July 1, 1948.

13. Marcum, J.I., "Tables of Q-Functions," Rand Corporation Report RM-339, Jan. 1, 1950.

14. Stewart, J.L., and E.C. Westerfield, "A Theory of Active Sonar Detection," Proc. I.R.E., 872-881, May 1959.

15. Westerfield, E.C., R.H. Prager, and J.L. Stewart, "Processing Gains Against Reverberation (Clutter) Using Matched Filters," Trans. I.R.E. on Information Theory, IT-6, 342-348, June 1960.

16. Green, P.E., "Radar Astronomy Techniques," Technical Report No. 282, M.I.T. Lincoln Laboratory.

17. Siebert, W.M., "Studies of Woodward's Uncertainty Function," Quarterly Progress Report, Research Laboratory of Electronics, M.I.T., 90-94, April 15, 1958.

18. Erdelei, H., Tables of Higher Transcendental Functions.

19. Price, R., and P.E. Green, Jr., "Signal Processing in Radar Astronomy-Communication via Fluctuating Multipath Media," Technical Report No.234, M.I.T. Lincoln Laboratory, October 1960.

20. Cramer, H., Mathematical Methods of Statistics, Princeton University Press, Princeton, N.J., 1946.

GLOSSARY

A	attenuation of target return
a	length parameter of Gaussian pulse
$a(t)$	intensity functions; average number of scatterers per unit time
B	rms Doppler shift (reverberation)
b	frequency parameter of Gaussian pulse
d_c^2	S/N ratio; conventional receiver
d_o^2	S/N ratio; optimum receiver
E_r	received energy
E_t	transmitted energy
$\tilde{f}(t)$	complex envelope $(\tilde{f}(t) = u(t)e^{j\phi(t)})$
$H_n(x)$	n^{th} Hermite function
$I_o(\cdot)$	Modified Bessel function; first kind; order zero
k	amplitude
L	rms length of scattering function
$M_{\nu_D}(ju)$	characteristic function of scatterer Doppler shift
$M_{\nu_q x}$	characteristic function of scatterer velocity ν_q is a random variable, x is conditioning variable
$\frac{N_o}{2}$	height of ambient noise spectral density (double-sided)
$N_A(t)$	actual additive noise

$\tilde{n}_A(t)$	complex envelope of additive noise
$N_r(t)$	actual reverberation return
$n_r(t)$	complex envelope of reverberation return
P_D	probability of detection
P_F	probability of false alarm
$p_{\omega_D}(x)$	probable density of scatterer Doppler shift
$Q(a,b)$	Marcum's Q function
$R(t)$	actual returned signal
$\tilde{r}(t)$	complex envelope of returned signal
Re	real part
$\tilde{S}_d(t)$	complex envelope of desired signal
$S_d(\omega)$	Fourier transform of $\tilde{S}_d(t)$
$S_n(t)$	complex envelope returned from n^{th} scatterer
$S_r(\omega)$	Fourier transform of $\tilde{R}(t)$
$S_T(t)$	transmitted signal
$S(v_1, v_2)$	characteristic function of joint scattering density
t_n	delay due to n^{th} scatterer
$u(t)$	actual envelope
Z_n	complex number which is the magnitude and phase of return
β	phase shift of target return
γ	scatterers per unit time (uniform case)
Δ	$\frac{a^2 + b^2}{a}$

Λ	likelihood ratio
Λ_o	a value of likelihood ratio
λ_n	n^{th} eigenvalue
$\phi(t)$	phase of transmitted signal
$\phi_n(t)$	n^{th} eigenfunction
$\theta(\tau,\omega)$	two-dimensional correlation factor
τ_D	target range (delay in signal return)
ω_c	carrier frequency
ω_D	target Doppler shift

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